

# STRATEGY-PROOFNESS AND SINGLE PEAKEDNESS IN BOUNDED DISTRIBUTIVE LATTICES

ERNESTO SAVAGLIO<sup>‡</sup> AND STEFANO VANNUCCI<sup>‡</sup>,

**ABSTRACT.** Two distinct specifications of single peakedness as currently met in the relevant literature are singled out and discussed. Then, it is shown that, under both of those specifications, a voting rule as defined on a bounded distributive lattice is strategy-proof on the set of all profiles of single peaked total preorders if and only if it can be represented as an iterated *median* of projections and constants, or equivalently as the behaviour of a certain median tree-automaton. The equivalence of individual and coalitional strategy-proofness that is known to hold for single peaked domains in bounded linear orders fails in such a general setting. A related impossibility result on anonymous coalitionally strategy-proof voting rules is also obtained.

*Keywords:* Strategy-proofness, Single Peakedness, Bounded Distributive Lattice, Voting Rule, Median.

MSC 2010 Classification: 05C05, 52021, 52037

JEL Classification Number: D71

## 1. INTRODUCTION

A good decision on an issue of social concern such as location of a public facility or choice of a tax rate has to rely on some information that is typically private and disperse among the relevant stakeholders. In many cases, voting by a suitable committee is one of the most practical means to elicit and amalgamate such information, and produce the final decision. Since part of the relevant information is private, however, voters may attempt to manipulate the outcome by misrepresenting that information (say, their most preferred outcome), and they are likely to do that *if* the voting rule allows for profitable *individual* manipulations. Now, that manipulative behaviour may easily result in inefficient outcomes, especially if enacted by several uncoordinated voters. Furthermore, the perceived availability of profitable manipulations may encourage diversion of voters' resources to gather private information concerning other voters. Thus, in order to prevent such possibly wasteful manipulative activities a voting rule should be reputedly *strategy-proof*, namely immune to advantageous individual manipulations. If, moreover, voters have access to cheap communication facilities allowing them to coordinate their voting strategies, then they can engage in coalitional manipulations as well. Therefore, in that case a voting rule should also be reputedly *coalitionally strategy-proof*, namely immune to jointly profitable manipulations on the part of *coalitions* of voters.

Those observations raise the following general identification issues:

- (i) *what are (if any) the strategy-proof voting rules on the relevant preference domain?*
- (ii) *which of them (if any) are also coalitionally strategy-proof?*

Of course, several alternative domains may be - and have been - taken into consideration in the aftermath of the Gibbard-Satterthwaite impossibility theorem. The present paper will address the

foregoing issues mainly focussing on an important class of *single peaked*<sup>1</sup> domains of total preorders in *bounded distributive lattices*<sup>2</sup>, namely the domains denoted here as *unimodal* and *locally strictly unimodal* (to be defined below). *A characterization of the entire class of strategy-proof voting rules on the full unimodal and locally strictly unimodal domains will be provided, generalizing or extending virtually all previously known results of that kind.* Quite remarkably, *the simple majority rule or extended median* (that is well-known to be strategy-proof and coalitionally strategy-proof on both unimodal and locally strictly unimodal domains in bounded chains) *is confirmed to belong to the strategy-proof class even in the present wider setting.* On the other hand, *it will also be shown that in a very large class of bounded distributive lattices that are not linear orders, and under minimal neutrality*<sup>3</sup> *requirements, no anonymous voting rule is coalitionally strategy-proof on the foregoing domains.*

Single peaked preferences arise in a natural way whenever each agent's representation of the outcome space is endowed with some 'natural' ternary *betweenness* relation establishing for any two outcomes  $x, y$  whether an arbitrary outcome  $z$  lies between  $x$  and  $y$  or not: indeed, single peaked total preorders are those total preorders with a *unique best outcome* that *respect -are consistent with- such betweenness relation.*

However, such a broad description of single peakedness is in fact compatible with *several distinct* specifications of the domain of single peaked preference relations.

At least two salient issues require further preliminary clarification, namely:

- (a) is the relevant betweenness relation agent-invariant (hence unique) or agent-dependent, and
- (b) what is precisely meant by 'consistency of preferences with the relevant betweenness relations'?

Concerning the first issue, the present paper follows the tradition that can be traced back at least to Black (1948) and was largely taken for granted in the early social choice theoretic literature: *the relevant betweenness relation is required to be agent-invariant hence unique across voters*, modeling a representation of the outcome structure that is entirely *shared* by all the involved parties.

Concerning the second issue, we focus on the two main variants encountered in the literature on single peakedness, namely

(1) the '*compromise*'-view of *betweenness-consistency* for preferences: *if an outcome is intermediate between two outcomes  $x$  and  $y$  then it is to be regarded as a 'compromise' between those two locally 'extreme' outcomes and as such is not strictly worse than both  $x$  and  $y$ . Such a 'compromise'-view can also be given a 'proximity'-interpretation, relying on a suitable metric induced by the underlying lattice. That notion of betweenness-consistency is in fact, arguably, the most 'natural' and appropriate one whenever the outcome set is a distributive lattice;*

(2) an alternative '*top proximity*'-view of *betweenness-consistency* for preferences: *if an outcome is intermediate between the top outcome and another outcome  $y$  and distinct from the latter, it is also closer than  $y$  to the top outcome and therefore strictly better than  $y$ .* It turns out, however,

---

<sup>1</sup>'Single peakedness' will be used as a general non-technical term that admits of several specifications, including of course unimodality and locally strict unimodality as introduced below.

<sup>2</sup>A distributive lattice is a partially ordered set such that any two elements  $x, y$  admit a least upper bound or join  $x \vee y$  and a greatest lower bound or meet  $x \wedge y$  that mutually 'distribute' on each other i.e. interact much like set-theoretic union and intersection. Clearly, the join and meet of any finite set of elements are also well-defined.

<sup>3</sup>A voting rule is neutral with respect to a certain pair of outcomes when it treats them in an unbiased manner.

that *such a notion of ‘top proximity’* -when employed in a distributive lattice that is not a chain, and defined through a suitable latticial metric (as discussed below under Remark 1)- generates a somewhat spurious notion of local single-peakedness allowing for the existence of *many local peaks* (more on this point below).

For the sake of convenience we shall denote as *unimodal* (*locally strictly unimodal*, respectively) precisely those preference profiles of total preorders that are *single peaked under specification (1) ((2), respectively)* of betweenness-consistency. Indeed, most contributions in the literature on single peakedness and strategy-proofness of voting rules and social choice functions focus exactly on unimodal or locally strictly unimodal preference profiles as defined above.

In a pioneering paper, Moulin (1980) characterizes the class of all strategy-proof voting rules (or, equivalently, ‘top-only’ social choice functions) on the domain of all total preorders that are unimodal with respect to the ‘natural’ betweenness relation of a bounded linearly ordered outcome set. In fact, he shows that such strategy-proof voting rules are precisely those based on the median as applied to voters’ choices possibly augmented with a certain set of fixed outcomes aptly dubbed ‘phantom vote(r)s’ by Border and Jordan (1983). Moreover, in the foregoing work Moulin points out that *those voting rules are also coalitionally strategy-proof*.

In a remarkable subsequent contribution, Danilov (1994) provides a similar median-based characterization of strategy-proof voting rules on the domain of all linear preference orders that are unimodal with respect to the ‘natural’ betweenness relation of a (bounded) undirected *tree*, and establishes *equivalence of individual and coalitional strategy-proofness* on that domain.

Now, (bounded) linear orders or chains are a quite special subclass of (bounded) distributive lattices, and many outcome spaces which are of interest for multi-agent aggregation problems and are not chains do share with chains precisely that latticial structure. Several examples of such outcome spaces will be presented and discussed in some detail in Section 3 below. They include spaces consisting respectively of *choice functions or generalized revealed preference relations, graded evaluations of items, systems of poverty thresholds, judgments consisting of deductively closed sets of statements, binary (dis)similarity relations or matrices, portfolios of basic derivative assets*.

Moreover, a *very natural betweenness relation is available in any lattice*: just declare  $z$  to lie *between*  $x$  and  $y$  if  $z$  is larger than -or equal to- the meet of  $x$  and  $y$  (written  $x \wedge y$ ) and smaller than -or equal to- the join of  $x$  and  $y$  (written  $x \vee y$ ).<sup>4</sup> Hence, both unimodal and locally strictly unimodal preference domains of total preorders can be easily defined for all (bounded) distributive lattices with respect to the standard latticial betweenness, namely: *a total preference preorder on a distributive lattice is unimodal if it has a unique maximum and is ‘compromise’-consistent (‘top proximity-consistent’, respectively) with the latticial betweenness relation*.

No characterizations of strategy-proof voting rules on unimodal domains in *general (possibly infinite) bounded distributive lattices other than chains* are available in the extant literature. In particular, the equivalence issue concerning strategy-proofness and coalitional strategy-proofness of voting rules on unimodal domains has never been addressed before in the foregoing latticial setting.

---

<sup>4</sup>That is, for instance, the notion of betweenness underlying ‘intermediate preferences’ as introduced by Grandmont (1978) and recently reconsidered by Bossert and Sprumont (2014) in their study of strategy-proof preference aggregation rules. That is also the betweenness relation underlying the proximity-based interpretation of single peakedness used by Barberà, Gul and Stacchetti (1993) who rely on the  $L_1$ -metric (or ‘taxi-cab’ metric).

To be sure, there are some partial results implying existence of strategy-proof voting rules but not of coalitionally strategy-proof voting rules on *locally strictly unimodal* domains in some *special* distributive lattices. But those results *do not address at all the case of unimodal domains in bounded distributive lattices* since they variously concern top-proximity-based single peaked preferences with respect to Euclidean-metric-betweenness in  $m$ -dimensional Euclidean spaces <sup>5</sup> with  $m \geq 2$  (see e.g. Border and Jordan (1983), Bordes, Laffond and Le Breton (2012)), or preference domains such as *locally strictly unimodal domains in finite products of bounded chains or in finite distributive lattices* (Barberá, Gul and Stacchetti (1993) and (Nehring and Puppe (2007 a, b), respectively) or *separable preferences* (Barberá, Sonnenschein and Zhou (1991)) *in certain finite Boolean lattices of cardinality eight or more*<sup>6</sup>. The latter preference domains, however, turn out to be *disjoint from unimodal domains on that class of lattices*. Moreover, the foregoing works do not address at all the general case of bounded distributive lattices.

But then, *what about strategy-proof voting rules on unimodal (or locally strictly unimodal) domains in arbitrary bounded distributive lattices? Are they median-representable? When do they also enjoy equivalence of individual and coalitional strategy-proofness?*

The present paper aims at filling this significant gap in the literature and provides a study of strategy-proofness and unimodality in *general* bounded distributive lattices. A median-based characterization of strategy-proof voting rules on unimodal and locally strictly unimodal domains in bounded distributive lattices is established by introducing *median tree-automata representations of voting rules*. It is a remarkable feature of our characterization that it unifies (generalizing or extending, and bringing together) several notions, approaches and results from the extant literature, namely:

- The characterization contributed by the present paper generalizes Moulin's original characterization of strategy-proof voting rules on unimodal domains in bounded chains to both (full) unimodal and locally strictly unimodal domains in *all* bounded distributive lattices, obtaining a lattice-polynomial representation which is a dual -and equivalent- version of that produced by the former author for the special case of bounded linear orders or chains (Moulin (1980)).
- Our characterization also highlights the equivalence of that lattice-polynomial representation to another and new representation of strategy-proof voting rules on unimodal domains as the *behaviour maps* of certain *median tree-automata* acting on suitably labelled trees. That tree-automata-theoretic representation essentially amounts to a streamlining and extension of the approach pioneered by Danilov (1994) in his remarkable characterization of strategy-proof voting rules on unimodal domains of linear orders in bounded trees via an interval-monotonicity property.
- The lattice-polynomial representation mentioned above is in turn a generalization of 'latticial-federation consensus functions' or, equivalently, of 'generalized committee voting rules' as introduced respectively, and independently, by Monjardet (1990) in his path-breaking contribution to (non-strategic) aggregation problems in latticial structures, and by Barberá,

<sup>5</sup>Recall that Euclidean spaces may be regarded as Riesz spaces i.e. as partially ordered vector spaces that are also (distributive) lattices.

<sup>6</sup>A Boolean lattice is a bounded distributive lattice with upper bound 1 and lower bound 0 such that each element  $x$  has a *complement*  $x'$  satisfying both  $x \vee x' = 1$  and  $x \wedge x' = 0$ .

Sonnenschein and Zhou (1991) in their well-known study of strategy-proof voting mechanisms on *separable* preference domains in finite Boolean lattices.

- Finally, it is also proved that the equivalence between strategy-proofness and coalitional strategy-proofness - that is known to hold for both unimodal and locally strictly unimodal domains in bounded linear orders and for unimodal domains of linear orders in bounded trees- *fails* for both unimodal and locally strictly unimodal domains in bounded distributive lattices that are *not* linear orders,<sup>7</sup> hence *even in outcome spaces with a well-defined (and unique) median operation* (Theorem 2). An impossibility theorem concerning coalitional strategy-proofness on the full unimodal and locally strictly unimodal domains for anonymous voting rules satisfying very weak local sovereignty and neutrality requirements (Theorem 3) is also provided. Thus, in particular, Theorem 3 establishes that the former equivalence may fail in bounded distributive lattices *even for non-sovereign voting rules*<sup>8</sup>.

The remainder of the paper is organized as follows. The next section describes several remarkable examples of bounded distributive lattices that occur in some well-known aggregation problems. Section 3 introduces the notation and definitions and includes the main results of the paper on the structure of the strategy-proof voting rules for full unimodal and locally strictly unimodal domains of total preorders in arbitrary bounded distributive lattices. In Section 4 the main results of the present work are discussed in some detail with reference to a simple example concerning the Boolean square. Section 5 includes a detailed discussion of some related literature and offers some concluding remarks. Appendix 1 collects all the proofs. Appendix 2 is devoted to a detailed presentation of the basic notions on tree automata used in the paper.

## 2. WHAT KIND(S) OF SINGLE PEAKEDNESS AND WHY BOUNDED DISTRIBUTIVE LATTICES?

Thus, our analysis shall be focussed on strategy-proof voting rules for both *unimodal and locally strictly unimodal* preference profiles in bounded distributive lattices as informally defined in the Introduction. Therefore, the model to be introduced below applies under the following conditions:

(1) the outcome set is a partially ordered set  $\mathcal{X} = (X, \leq)$  with a top and a bottom, and such that the (binary) least-upper bound  $\vee$  and greatest-lower bound  $\wedge$  as induced by  $\leq$  satisfy the distributive identity  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$  for any  $x, y, z \in X$ ;

(2) all voters are prepared to assess outcomes according to the latticial ternary betweenness relation of  $\mathcal{X}$  denoted  $B_{\mathcal{X}}$  : namely, outcome  $z$  lies between outcomes  $x$  and  $y$  if and only if  $x \wedge y \leq z \leq x \vee y$  ;

(3) the preferences of all voters are consistent with the latticial betweenness relation in one of the two following senses: (i) (unimodality) any outcome that  $z$  lies between outcomes  $x$  and  $y$  is to be regarded as a ‘compromise’ between its relative extrema  $x$  and  $y$  and is therefore not strictly

<sup>7</sup>Vannucci (2012) provides a general incidence-geometric argument to explain that equivalence-failure.

<sup>8</sup>Indeed, Nehring and Puppe (2007 (b)) prove that the only efficient and strategy-proof voting rules on certain ‘rich’ subdomains of locally strictly unimodal domains of linear orders in finite Boolean lattices or  $m$ -hypercubes  $2^m$  with  $m \geq 3$  are (weakly) dictatorial. Notice, however, that such preference domain is incomparable to our unimodal domain and (weakly) efficient voting and social choice rules are in particular sovereign: thus, Nehring and Puppe’s result pertains to a class of rules which is utterly non-comparable to the class of voting rules covered by Theorem 3 of the present work.

worse than each one of the latter; (ii) (locally strict unimodality) any outcome that  $z$  lies between top outcome  $x$  and outcome  $y$  and is distinct from  $y$  is to be regarded as ‘closer’ to the top outcome than  $y$  and is therefore strictly better than  $y$ .

Observe that when specialized to the particular case of a bounded chain, conditions (1)-(2)-(3(i)) (or (1)-(2)-(3(ii))) are a list of plausible requirements to be met in order to justify single peaked domains in the standard version pioneered by Black (1948). Indeed, the foregoing conditions - especially (1)-(2)-(3(i)) - arguably provide the ‘right’ extension of the notion of a single peaked preference *profile* for distributive lattices as illustrated in the simple example concerning Boolean squares as discussed above. Of course, even if condition (1) obtains, conditions (2) and/or (3) may or may not hold depending on the problem under scrutiny. But it is at least conceivable that there are interesting cases where conditions (1)-(2) and either (3(i)) or (3(ii)) are jointly satisfied, and when that is the case, focusing on the full unimodal (or locally strictly unimodal) preference domain of a bounded distributive lattice as defined in the present work seems to be fully justified, indeed somewhat compelling.

In order to fully appreciate the remarkably wide scope and relevance of the proposed setting let us consider just a few prominent classes of examples of bounded distributive lattices of special interest, namely:

**Example 1: Committee decision on multidimensional binary issues: aggregation of points on a finite Boolean hypercube.**

Let  $\mathbf{2}^k$  be the set of points of a finite  $k$ -dimensional Boolean hypercube and  $\leq$  the standard componentwise order. Then, take the  $\mathcal{X} = (\mathbf{2}^k, \leq)$ . When considering an abstract location problem on that discrete cube such as committee-selection of the *appropriate profile of binary criteria to be satisfied by candidates* in order to qualify for a certain position: here, one has to face the issue of aggregating the alternative proposals (namely points of  $\mathbf{2}^k$ ) advanced by members of a panel committee.

**Example 2: Committee selection of location on a multidimensional box: aggregation of points in a product of bounded subsets of the extended real line.**

Let  $\mathbb{R}_* = \mathbb{R} \cup \{-\infty, +\infty\}$  denote the extended real line,  $\leq^*$  the component-wise extended natural order on  $\mathbb{R}_*^m$ ,  $Y_i \subseteq \mathbb{R}_*$  for each  $i = 1, \dots, m$ , and  $x, y \in \prod_{i=1}^m Y_i$  with  $x \leq^* y$ . Then, take  $\mathcal{X} = (X, \leq)$

with  $X = \left\{ z \in \prod_{i=1}^m Y_i : x \leq^* z \leq^* y \right\}$  and  $\leq = \leq|_X^*$  (recall that a product of distributive lattices is a distributive lattice under the component-wise order). This is the setting of Barberà, Gul and Stacchetti (1993) study of strategy-proofness on locally strictly unimodal domains. If  $m = 1$ ,  $(X, \leq)$  reduces to a bounded chain, which gives the original standard setting of the literature on strategy-proofness on single peaked domains, including the seminal work of Moulin (1980) on the characterization of strategy-proof voting rules on unimodal domains in a bounded real chain, where  $\mathcal{X} = (\mathbb{R}_*, \leq^*)$ .

**Example 3: Election of a representative body and committee selection of the target of an ‘atomic’ package bid in a combinatorial auction: aggregation of subsets of a fixed set.**

Let  $Y$  be a set of items,  $\mathcal{P}(Y)$  its power set and  $\Sigma \subseteq \mathcal{P}(Y)$  a field of sets (namely  $\Sigma$  is nonempty and such that  $\{A \cap B, A \cup B, X \setminus A\} \subseteq \Sigma$  for any  $A, B \in \Sigma$ ). Then, take  $\mathcal{X} = (\Sigma, \subseteq)$ . This kind of domain arises in a most natural way in a few cases including *combinatorial* social choice problems, i.e. social choice issues concerning *mutually compatible* objects (e.g. selection by committee decision of a representative body, or of the target of an admissible package bid in a combinatorial auction namely the item-subset to bid for: in the latter case,  $\Sigma$  denotes the set of admissible packages as fixed by the auction mechanism designer with a view to keep communication complexity under some acceptable threshold).

**Example 4: Committee selection of a portfolio of basic derivative assets: aggregation of points in a bounded Riesz space.**

Let  $s : [0, 1] \rightarrow \mathbb{R}_+$  be a continuous non-negative real-valued function denoting a limited-liability state-dependent stock with state-space  $[0, 1]$ , and  $b : [0, 1] \rightarrow \mathbb{R}_+$  a constant function denoting the relevant bond; if  $s([0, 1]) = [\alpha, \beta]$  then ordered linear space  $(X = \mathcal{C}[\alpha, \beta], \leq)$ , the set of continuous real-valued functions on  $[\alpha, \beta]$  endowed with the component-wise natural order, denotes the space of all *continuous options on  $s$* . It turns out that  $(\mathcal{C}[\alpha, \beta], \leq)$  is in fact a *Riesz space* namely it is also a (distributive) lattice: moreover, it is *bounded* with constant functions  $f_\alpha$  and  $f_\beta$  as bottom and top elements, respectively. In particular, the latticial operations  $\vee$  and  $\wedge$  of  $(\mathcal{C}[\alpha, \beta], \leq)$  enable convenient representations of both call options on  $s$  and put options on  $s$  at any striking price  $p$  as  $(s - pb) \vee 0$  and  $(pb - s) \vee 0$ , respectively, and it can be shown that each continuous option in  $\mathcal{C}[\alpha, \beta]$  can be represented as a *portfolio of call options* (see e.g. Brown and Ross (1991)). Thus, under the foregoing stipulations a committee selection of a continuous option on a stock (or equivalently of a portfolio of call options on that stock) amounts to an aggregation of points in bounded Riesz space  $\mathcal{X} = (\mathcal{C}[\alpha, \beta], \leq)$ .

**Example 5: General revealed preference aggregation: aggregation of choice functions on a fixed set.**

Let  $Y$  be a set of items, and  $\mathcal{P}(Y)$  its power set. A (full-domain) *choice function* on  $Y$  is a function  $f : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$  such that  $f(A) \subseteq A$  for each  $A \subseteq Y$ . Now, denote by  $\mathcal{C}_Y$  the set of all choice functions on  $Y$ , and for any  $f, g \in \mathcal{C}_Y$  posit  $f \leq' g$  if and only if  $f(A) \subseteq g(A)$  for all  $A \subseteq Y$ . Then, take  $\mathcal{X} = (\mathcal{C}_Y, \leq')$ , where the constant empty-valued choice function is the bottom and the identity choice function is the top. The aggregation of the choice functions in  $\mathcal{C}_Y$  may be regarded as a *natural generalization of the classic problem of preference aggregation in social welfare analysis* if preferences are taken to summarize choice behaviour and the usual ‘consistency’ requirements related to acyclicity properties are relaxed. Indeed, the issue here is the elicitation and aggregation of complete lists of recommendations concerning local choice behaviour from a population of experts and/or stakeholders.

**Example 6: Merging databases of binary (dis)similarity coefficients: aggregation of dissimilarity and tolerance relations on a fixed set.**

Let  $Y$  a set of items: a *dissimilarity* (or orthogonality) relation on  $Y$  is an irreflexive and symmetric binary relation  $D$  on  $Y$  i.e.  $D \subseteq Y \times Y$  is such that (i)  $(y, y) \notin D$  for all  $y \in Y$  and (ii)  $(y, z) \in D$  implies  $(z, y) \in D$  for all  $y, z \in Y$ . Denote by  $\mathcal{D}_Y$  the set of all dissimilarity relations on  $Y$ , and take  $\mathcal{X} = (\mathcal{D}_Y, \subseteq)$ ; a *tolerance* (or similarity) relation on  $Y$  is a reflexive and symmetric binary relation  $D$  on  $Y$  i.e.  $D \subseteq Y \times Y$  is such that (i)  $(y, y) \in D$  for all  $y \in Y$  and (ii)  $(y, z) \in D$  implies  $(z, y) \in D$  for all  $y, z \in Y$ . Denote by  $\mathcal{T}_Y$  the set of all tolerance relations on  $Y$ , and take  $\mathcal{X} = (\mathcal{T}_Y, \subseteq)$ . Dissimilarity and tolerance relations are one of the basic inputs in most algorithmic classification procedures: if many (binary) dissimilarity databases from several distinct sources are available, one may wish to aggregate them to produce a *unique consensus database*.

**Example 7: Merging judgments with their implications: aggregation of order filters over a partially ordered set.**

Let  $\mathcal{Y} = (Y, \leq)$  denote a finite partially ordered population. An *order filter* of  $\mathcal{Y}$  is a set  $F \subseteq Y$  such that for all  $y, z \in Y$ ,  $z \in F$  whenever  $y \in F$  and  $y \leq z$ . Denote by  $\mathcal{F}_Y$  the set of all order filters of  $\mathcal{Y}$ , and take  $\mathcal{X} = (\mathcal{F}_Y, \subseteq)$ . Order filters may variously arise in several aggregation problems, including *judgment aggregation problems with implication-constrained agendas*. Thus, in the latter case  $Y$  denotes a collection of propositions (namely, sets of logically equivalent sentences) and  $\leq$  denotes the relevant *implication* or consequence relation between propositions. In that connection, a judgment amounts to a deductively closed set of propositions, namely an order filter of  $\mathcal{Y}$ , and establishing a *consensus judgment* reduces to aggregating order filters of  $\mathcal{Y}$ .

**Example 8: Merging proposals for multidimensional poverty thresholds: aggregation of order ideals over a partially ordered set.**

An *order ideal* of a partially ordered set  $\mathcal{Y} = (Y, \leq)$  is a set  $I \subseteq Y$  such that for all  $y, z \in Y$ ,  $z \in I$  whenever  $y \in I$  and  $z \leq y$ . Denote by  $\mathcal{I}_Y$  the set of all order ideals of  $\mathcal{Y}$ , and take  $\mathcal{X} = (\mathcal{I}_Y, \subseteq)$ . Order ideals are also relevant to several aggregation problems, including choice of a (system of) *threshold(s)* in *multidimensional poverty analysis*: a list of relevant binary attributes is considered, and different thresholds namely combinations of minimal deprivations are proposed by qualified experts and/or political representatives to identify the poor. Each threshold corresponds to an order ideal, hence amalgamating the advanced proposals amounts to aggregating order ideals.

**Example 9: Merging graded assessments, and computing reputations: aggregation of graded evaluations.**

Let  $\Lambda = (L, \leq)$  denote a (bounded) linearly ordered set of grades,  $X$  a (finite) population of candidates to be evaluated, and  $N$  a (finite) population of evaluators. Then, denote by  $L^X$  the set of all possible gradings of  $X$ , by  $\leq$  the point-wise partial order induced by  $\leq$ , and take  $\mathcal{X} = (L^X, \leq)$ . This is indeed the formal setting recently proposed by Balinski and Laraki (2010) in order to advance their case for *majority judgment*. It may be considered for aggregating grades achieved by a population of students in different subjects, assessments of wines according to several alternative graded criteria or the graded performances of participants in a multi-trial competition. A particular case of special interest is provided by the definition of reputation systems, both off-line and on-line. Indeed, by taking  $L \subseteq \mathbb{Z}$  (or  $L \subseteq \mathbb{Q}$ ) with the natural order of the integers (or the rationals), and



$N = X$ , a point of  $L^N$  denotes the approval/citation profile of an author (or the on-line feedback profile of a web-node), and a *reputation system* is an aggregation function  $f : (L^N)^N \rightarrow L^N$ .

The foregoing set of examples is of course not meant to be an exhaustive list, and some of them may well refer to comparatively more uncommon or hypothetical decision problems than others. However, that list provides in our view a quite representative sample of the wide class of interesting aggregation problems to which our results on strategy-proof voting rules in bounded distributive lattices do in fact apply.

### 3. MODEL AND RESULTS

Let  $N = \{1, \dots, n\}$  denote the finite population of voters, and  $\mathcal{X} = (X, \leq)$  the partially ordered set of alternative outcomes (i.e.  $\leq$  is a reflexive, transitive and antisymmetric binary relation on  $X$ ). We suppose  $|N| \geq 3$  in order to avoid tedious qualifications, and denote as  $x \parallel y$  any pair of  $\leq$ -incomparable outcomes.<sup>9</sup> Let us also assume that  $\mathcal{X} = (X, \leq)$  is a **distributive lattice** namely both the *least-upper-bound* (l.u.b.)  $\wedge$  and the *greatest-lower-bound* (g.l.b.)  $\vee$  of any  $x, y \in X$  as induced by  $\leq$  are well-defined binay operations on  $X$ , and for all  $x, y, z \in X$ ,  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$  (or, equivalently,  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ ).<sup>10</sup> In particular,  $\mathcal{X} = (X, \leq)$  is a **linear order** or **chain** if  $[x \leq y \text{ or } y \leq x]$  holds for all  $x, y \in X$  (recall that, as it is easily checked, a chain does indeed satisfy the distributive identity above). A *join irreducible* element of  $\mathcal{X}$  is any  $j \in X$  such that  $j \neq \wedge X$  and for any  $Y \subseteq X$  if  $j = \vee Y$  then  $j \in Y$ . The set of all join irreducible elements of  $\mathcal{X}$  is denoted  $J_{\mathcal{X}}$ . An *atom* of a lower bounded  $\mathcal{X}$  is any  $\leq$ -minimal  $x \in X \setminus \{\perp\}$ . The set of all atoms of  $\mathcal{X}$  is denoted  $A_{\mathcal{X}}$ : clearly,  $A_{\mathcal{X}} \subseteq J_{\mathcal{X}}$ . Moreover, a (distributive) lattice  $\mathcal{X}$  is said to be **lower (upper) bounded** if there exists  $\perp \in X$  ( $\top \in X$ ) such that  $\perp \leq x$  ( $x \leq \top$ ) for all  $x \in X$ , and **bounded** if it is both lower bounded and upper bounded. A bounded distributive lattice  $(X, \leq)$  is **Boolean** if for each  $x \in X$  there exists a *complement* namely an  $x' \in X$  such that  $x \vee x' = \top$  and  $x \wedge x' = \perp$ . A ternary **betweenness** relation  $B_{\mathcal{X}} = \{(x, z, y) \in X^3 : x \wedge y \leq z \leq x \vee y\}$  is defined on  $\mathcal{X}$ , and  $x, y \in X$ ,  $[x, y] = \{z \in X : x \wedge y \leq z \leq x \vee y\}$  is the *interval* induced by  $x$  and  $y$ : therefore, for any  $x, y, z \in X$ ,  $z \in [x, y]$  if and only if  $(x, z, y) \in B_{\mathcal{X}}$  (also written  $B_{\mathcal{X}}(x, z, y)$ ).<sup>11</sup>

It is a remarkable fact that a ternary operation called *median* is well-defined on an arbitrary distributive lattice.

**Definition 1.** The *median* on  $\mathcal{X}$  is the ternary operation  $\mu : X^3 \rightarrow X$  defined as follows: for all  $x, y, z \in X$ ,

$$\mu(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z).$$

Notice that, due to commutativity and associativity of  $\wedge$  and  $\vee$  the median  $\mu$  as defined above is *invariant under permutations of its arguments* or *symmetric*, namely for any  $x_1, x_2, x_3 \in X$  and any permutation  $\sigma : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ ,  $\mu(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = \mu(x_1, x_2, x_3)$ .<sup>12</sup>

<sup>9</sup>We denote by  $|\cdot|$  the cardinality of a set.

<sup>10</sup>Notice that thanks to associativity of  $\vee$  and  $\wedge$  the l.u.b. and the g.l.b. of any finite  $Y \subseteq X$  are also well-defined and denoted by  $\vee Y$  and  $\wedge Y$ , respectively; if  $Y$  is infinite  $\vee Y$  and  $\wedge Y$  may or may not be well-defined.

<sup>11</sup>The ensuing analysis could be pursued by replacing entirely betweenness relations with intervals (see Vannucci (2012) for such an approach in a more general setting).

<sup>12</sup>It should be recalled here that Birkhoff and Kiss (1947) also provide a general characterization of the median in a (bounded) distributive lattice through the following axioms for an arbitrary ternary operation  $m$  on a set  $A$ :

**Remark 1.** Notice that a natural median-based betweenness relation  $B_{\mathcal{X}}^{\mu} \subseteq X^3$  can be defined on  $\mathcal{X}$  by the following rule: for any  $x, y, z \in X$ ,  $(x, z, y) \in B_{\mathcal{X}}^{\mu}$  iff  $\mu(x, y, z) = z$ . But it is easily shown that in fact  $B_{\mathcal{X}}^{\mu} = B_{\mathcal{X}}$  (see Birkhoff and Kiss (1947), Theorem 1). Moreover, if the relevant distributive lattice  $(X, \leq)$  is metric i.e. is endowed with a positive valuation namely a function  $v : X \rightarrow \mathbb{R}$  such that  $v(x \vee y) = v(x) + v(y) - v(x \wedge y)$  and  $v(x) < v(y)$  whenever  $x < y$ , then it can be shown that  $B_{\mathcal{X}} = \{(x, y, z) \in X^3 : d_v(x, z) = d_v(x, y) + d_v(y, z)\}$ , where  $d_v$  is the metric induced by  $v$  as defined by the rule  $d_v(x, y) = v(x \vee y) - v(x \wedge y)$  (see Glivenko (1936), Theorem V). Thus  $B_{\mathcal{X}}(x, y, z)$  holds precisely when  $y$  lies on a  $d_v$ -geodesics or  $d_v$ -shortest path joining  $x$  and  $z$ . That fact suggests the possibility to provide  $B_{\mathcal{X}}$  with a straightforward metric representation whenever deemed appropriate, and highlights the focal role of  $B_{\mathcal{X}}$  and of the median operation for any plausible proximity relation respecting the latticial structure of  $\mathcal{X}$ .

A few remarkable basic properties of  $B_{\mathcal{X}}$  are listed below:

**Claim 1.** The latticial betweenness relation  $B_{\mathcal{X}}$  satisfies the following conditions:

- (i) symmetry: for all  $x, y, z \in X$ , if  $B_{\mathcal{X}}(x, z, y)$  then  $B_{\mathcal{X}}(y, z, x)$ ;
- (ii) closure (or reflexivity): for all  $x, y \in X$ ,  $B_{\mathcal{X}}(x, x, y)$  and  $B_{\mathcal{X}}(x, y, y)$ ;
- (iii) idempotence: for all  $x, y \in X$ ,  $B_{\mathcal{X}}(x, y, x)$  only if  $y = x$ ;
- (iv) convexity (or transitivity): for all  $x, y, z, u, v \in X$ , if  $B_{\mathcal{X}}(x, u, y)$ ,  $B_{\mathcal{X}}(x, v, y)$  and  $B_{\mathcal{X}}(u, z, v)$  then  $B_{\mathcal{X}}(x, z, y)$ ;
- (v) antisymmetry: for all  $x, y, z \in X$ , if  $B_{\mathcal{X}}(x, y, z)$  and  $B_{\mathcal{X}}(y, x, z)$  then  $x = y$ .

Now, consider the set  $T_X$  of all *topped* total preorders on  $X$  (i.e. *connected*, reflexive, and transitive binary relations having a unique maximum in  $X$ ). For any  $\succsim \in T_X$ ,  $\text{top}(\succsim)$  denotes the unique maximum of  $\succsim$  (while  $\succ$  and  $\sim$  denote the asymmetric and symmetric components of  $\succsim$ , respectively).

**Definition 2.** A topped total preorder  $\succsim \in T_X$  is **unimodal** (with respect to  $B_{\mathcal{X}}$ ) if and only if, for each  $x, y, z \in X$ ,  $z \in [x, y]$  implies that either  $z \succsim x$  or  $z \succsim y$  (or both).

As mentioned above, the rationale underlying single peakedness as *unimodality* may be plainly described as follows: an unimodal total preference preorder *respects* betweenness  $B_{\mathcal{X}}$  in that it never regards an intermediate or compromise outcome as strictly worse than both of its ‘extreme’-generators.

An alternative notion of single peakedness has also been widely adopted in the literature under several labels including ‘generalized single peakedness’ (see e.g. Nehring and Puppe (2007 (a,b) among others)). It will be relabeled here ‘*locally strict unimodality*’ for the sake of convenience, and may be formulated as follows in the present setting:

**Definition 3.** A topped total preorder  $\succsim \in T_X$  (with top outcome  $x^*$ ) is **locally strictly unimodal** (with respect to  $B_{\mathcal{X}}$ ) if and only if, for each  $y, z \in X$ ,  $z \in [x^*, y] \setminus \{y\}$  implies  $z \succ y$ .

---

m(i) there exist  $0, 1 \in A$  such that  $m(0, a, 1) = a$  for all  $a \in A$ ;

m(ii)  $m(a, b, a) = a$  for all  $a, b \in A$ ;

m(iii)  $m(a, b, c) = m(b, a, c) = m(b, c, a)$  for all  $a, b, c \in A$ ;

m(iv)  $m(m(a, b, c), d, e) = m(m(a, d, e), b, m(c, d, e))$  for all  $a, b, c, d, e \in A$ .

An alternative characterization of the latticial median is provided by Sholander (1954(a)).

**Remark 2.** It is worth noticing here that in the extant literature unimodality and locally strict unimodality are not always firmly distinguished as they should be. For instance, in a very interesting and widely cited paper Nehring and Puppe (2007 (b), p.135) quote Moulin (1980) as a contribution on ‘generalized single peaked’(i.e. locally strictly unimodal) preferences in the case of a line (but see also Barberà, Gul and Stacchetti (1993) who identify single peakedness and locally strict unimodality, suggesting that this is precisely the notion underlying Moulin’s work). However, Moulin’s definition, once reformulated in terms of preferences (as opposed to utilities, as in the original Moulin (1980), p. 439) amounts to the following requirement: ‘If  $a$  is the top outcome or peak on the line  $(X, \leq)$  then  $a \succ x \succ y$  if  $a < x \leq y$  or  $y \leq x < a$ ’. Notice however that this condition is only consistent with *unimodality* as opposed to locally strict unimodality. To see this just consider  $X = \{a, x, y\}$  with  $a < x < y$ , and total preorder  $\succsim$  such that  $a \succ x \sim y$ : by construction,  $\succsim$  is certainly consistent with Moulin’s condition, and it is in fact unimodal but **not** at all locally strictly unimodal (or ‘generalized single peaked’).

By definition, existence of unimodal total preorders with the respect to latticial betweenness  $B_X$  on a lattice  $X = (X, \leq)$  is clearly not an issue: for any  $x \in X$ , the total preorder with  $x$  as its unique top outcome and  $y \sim z$  for any other  $y, z \in X$  is by construction unimodal. On the other hand, existence of locally strictly unimodal total preorders with respect to  $B_X$  requires a more detailed argument that relies on some specific properties of  $B_X$  as combined with the following notions of *Suzumura-consistency* (henceforth, *S-consistency*) and *non-trivial total extension* of a binary relation:

**Definition 4.** (a) A binary relation  $\succsim$  on  $X$  is **S-consistent** whenever for all  $x, y \in X$ , if for some positive integer  $k$  there exist  $z_1, \dots, z_k \in X$  such that  $x \succ z_1$ ,  $z_k \succ y$  and  $z_h \succ z_{h+1}$ ,  $h = 1, \dots, k-1$  then **not**  $y \succ x$ ;

(b) A binary relation  $\succsim'$  is a **non-trivial extension** of binary relation  $\succsim$  on  $X$  if for all  $x, y \in X$ : (i)  $x \succ y$  entails  $x \succ' y$ ; (ii)  $x \succ y$  entails  $x \succ' y$ .

The former notions are mutually related thanks to a generalization of Szpilrajn’ theorem on ordering extensions due to Suzumura, namely:

**Suzumura’s Theorem** (see e.g. Bossert and Suzumura (2010), Theorem 2.8) *A binary relation  $\succsim$  on a set  $X$  admits a total preorder as a non-trivial extension if and only if it is S-consistent.*

It turns out that the following claim can be quite easily established:

**Claim 2.** Let  $X = (X, \leq)$  be a (bounded) distributive lattice,  $B_X$  its (latticial) betweenness relation as defined above,  $x \in X$  and  $\succ_x$  the binary relation on  $X$  defined as follows: for all  $x, y, z \in X$ ,  $y \succ_x z$  if and only if  $B_X(x, y, z)$  and  $y \neq z$  (i.e.  $y \in [x, z] \setminus \{z\}$ ). Then  $\succ_x$  admits a non-trivial extension  $\succ_x^*$  which is a locally strictly unimodal total preorder on  $X$  with respect to  $B_X$ .

Let  $U_X \subseteq T_X$  denote the set of all unimodal total preorders (with respect to  $B_X$ ), and  $U_X^N$  the set of all *N-profiles* of unimodal total preorders or full unimodal domain (with respect to  $B_X$ ). Similarly,  $S_X \subseteq T_X$  is the set of all locally strictly unimodal total preorders (with respect to  $B_X$ ),

and  $S_{\mathcal{X}}^N$  denotes the set of all  $N$ -profiles of locally strictly unimodal total preorders or full locally strictly unimodal domain (with respect to  $B_{\mathcal{X}}$ ).

A *voting rule* for  $(N, X)$  is a function  $f : X^N \rightarrow X$ . For any profile  $(Y_i)_{i \in N}$  (where  $Y_i \subseteq X$  for all  $i \in N$ ) a *restricted voting rule* for  $(N, X)$  is a function  $f : \Pi_{i \in N} Y_i \rightarrow X$ . The following properties of a voting rule will play a crucial role in the ensuing analysis:

**Definition 5.** A voting rule  $f : \Pi_{i \in N} Y_i \rightarrow X$  is  $B_{\mathcal{X}}$ -**monotonic** if and only if for all  $x_N = (x_j)_{j \in N} \in Y^N$ ,  $i \in N$  and  $x'_i \in Y$ :  $f(x_N) \in [x_i, f(x'_i, x_{N \setminus \{i\}})]$ .

**Definition 6.** For any  $i \in N$ , let  $D_i \subseteq U_{\mathcal{X}}$  such that  $\text{top}(\succ) \in Y_i$  for all  $\succ \in D_i$ . Then,  $f : \Pi_{i \in N} Y_i \rightarrow X$  is (individually) **strategy-proof** on  $\Pi_{i \in N} D_i \subseteq U_{\mathcal{X}}^N$  if and only if, for all  $x_N \in \Pi_{i \in N} Y_i$ ,  $i \in N$  and  $x' \in Y_i$ , and for all  $\succ = (\succ_j)_{j \in N} \in \Pi_{i \in N} D_i$ ,  $f(\text{top}(\succ_i), x_{N \setminus \{i\}}) \succ_i f(x', x_{N \setminus \{i\}})$ .

**Definition 7.** For any  $i \in N$ , let  $D_i \subseteq U_{\mathcal{X}}$  such that  $\text{top}(\succ) \in Y_i$  for all  $\succ \in D_i$ . Then,  $f : \Pi_{i \in N} Y_i \rightarrow X$  is **coalitionally strategy-proof** on  $\Pi_{i \in N} D_i \subseteq U_{\mathcal{X}}^N$  if and only if for all  $x_N \in \Pi_{i \in N} Y_i$ ,  $C \subseteq N$  and  $x'_C \in \Pi_{i \in C} Y_i$ , and for all  $\succ = (\succ_j)_{j \in N} \in \Pi_{i \in N} D_i$ , there exists  $i \in C$  such that  $f(x_N) \succ_i f(x'_C, x_{N \setminus C})$ .

The following properties of voting rules will also be considered in the ensuing analysis.

A voting rule  $f : \Pi_{i \in N} Y_i \rightarrow X$  is (weakly) **efficient** if and only if for all  $(\succ_j)_{j \in N} \in \Pi_{i \in N} D_i \subseteq U_{\mathcal{X}}^N$  and  $y \in X$ ,  $y \notin f((\text{top}(\succ_j)_{j \in N}))$  if there exists  $x \in X$  such that  $x \succ_j y$  for all  $j \in N$ , **anonymous** if  $f((x_j)_{j \in N}) = f((x_{\sigma(j)})_{j \in N})$  for all  $x_N \in X^N$  and all permutations  $\sigma : N \rightarrow N$ , **locally JI-neutral** on  $Y \subseteq X$  if  $f((\tau_{jk}(x_i))_{i \in N}) = \tau_{jk}(f((x_i)_{i \in N}))$  for all  $y_N \in Y^N$  and  $j, k \in J_{\mathcal{X}} \cap Y$  (where  $\tau_{jk} : Y \rightarrow Y$  is the elementary permutation of  $Y$  such that  $\tau_{jk}(j) = k$ ,  $\tau_{jk}(k) = j$  and  $\tau_{jk}(x) = x$  for any  $x \neq j, k$ ), **locally sovereign** on  $Y \subseteq X$  if for all  $z \in Y$  there exists  $y_N \in Y^N$  such that  $f(y_N) = z$ , and **locally idempotent** on  $Y \subseteq X$  if  $f(y_N) = z$  for each  $y_N \in Y^N$  such that  $y_i = z$  for all  $i \in N$ .

A *generalized committee* in  $N$  is a set of coalitions  $\mathcal{C} \subseteq \mathcal{P}(N)$  such that  $T \in \mathcal{C}$  if  $T \subseteq N$  and  $S \subseteq T$  for some  $S \in \mathcal{C}$  (a *committee* in  $N$  being a *non-empty* generalized committee in  $N$  which does not include the *empty* coalition)<sup>13</sup>.

A **generalized committee voting rule** is a function  $f : \Pi_{i \in N} Y_i \rightarrow X$  such that, for some fixed generalized committee  $\mathcal{C} \subseteq \mathcal{P}(N)$  and for all  $y_N \in \Pi_{i \in N} Y_i$ ,  $f(y_N) = \bigvee_{S \in \mathcal{C}} (\bigwedge_{i \in S} x_i)$ .

A **generalized weak committee voting rule** is a function  $f : \Pi_{i \in N} Y_i \rightarrow X$  such that, for some fixed generalized committee  $\mathcal{C} \subseteq \mathcal{P}(N)$  and some fixed family  $\{z_S : z_S \in X\}_{S \in \mathcal{C}}$ , and for all  $y_N \in \Pi_{i \in N} Y_i$ ,  $f(y_N) = \bigvee_{S \in \mathcal{C}} ((\bigwedge_{i \in S} x_i) \wedge y_S)$ .

Two notable classes of strategy-proof voting rules are the *projections* (or *dictatorial rules*)  $\pi_i : Y^N \rightarrow X$ ,  $i \in N$  where for all  $y_N \in Y^N$ ,  $\pi_i(y_N) = y_i$ , and the *constant rules*  $f_x : Y^N \rightarrow X$ ,  $x \in X$  where for all  $y_N \in Y^N$ ,  $f_x(y_N) = x$ . It is also easily checked that *both dictatorial and constant rules are  $B_{\mathcal{X}}$ -monotonic*.<sup>14</sup>

The representation of  $B_{\mathcal{X}}$ -monotonic voting rules as *behaviour maps* of certain *tree automata* acting on suitably *labelled trees* will play a key role in the present paper. In that connection, a few

<sup>13</sup>Thus, a generalized committee is just an *order filter* of the partially ordered set  $(\mathcal{P}(N), \subseteq)$  of coalitions of  $N$ .

<sup>14</sup>Indeed, for all  $x_N = (x_j)_{j \in N} \in Y^N$ ,  $i \in N$  and  $x'_i \in Y$ :  $f(x_N) = x_i \in [x_i, f(x'_i, x_{N \setminus \{i\}})]$  if  $f$  is the  $i$ -th projection, and  $f(x_N) = f(x'_i, x_{N \setminus \{i\}}) \in [x_i, f(x'_i, x_{N \setminus \{i\}})]$  if  $f$  is a constant function.

supplementary definitions are to be introduced here concerning precisely certain tree automata and their behaviour (see Adámek and Trnková (1990) for a thorough treatment of tree automata).

A  $\Sigma$ -**tree automaton** is a general model of a mechanism that can perform certain *operations* on its ‘internal’ state space, and produce certain observable *outputs* as a response to certain *inputs* it is equipped to detect and act upon once it is suitably prepared to do so, or *initialized*. The *operations* a tree automaton is able to perform are algebraic or finitary (i.e. each of them applies to some fixed finite number of arguments, its ‘arity’), and are recorded together with their respective ‘arities’ by the automaton’s type, denoted  $\Sigma$ . The *inputs* of a tree automaton of a certain type  $\Sigma$  are finite trees with some terminal nodes labelled by variables of a certain set  $I$  that have to be initialized, whereas all the other terminal nodes are labelled by (symbols of) operations as recorded by type  $\Sigma$  (such trees are also denoted here as finite  $(\Sigma, I)$ -trees). The *initialization* of the automaton assigns one specific state to every variable of  $I$  and so makes it possible for the automaton to start its action on any suitable tree-input. A tree-input dictates the admissible (and mutually equivalent) sequences of operations to be performed by the automaton as it inspects its nodes moving backward from the terminal nodes. The final outcome of that sequence of operations or *run* of the initialized tree automaton produces as an end-result the state which is computed performing the operation labelled by the initial node or *root* of the tree as applied to the outcomes of its previous computations. The output function then works as an ‘effector’ that transforms the final state thus obtained into an observable output. The *behaviour* of a  $\Sigma$ -tree automaton denotes precisely the rule that transforms each  $(\Sigma, I)$ -tree-input into a certain observable output through the process just described (see Appendix 3 for a formal definition of  $\Sigma$ -tree automata and all the relevant details).

Actually, we shall be concerned with *lattice median -or l-median- tree automata*. A (non-initial) **l-median tree automaton**  $\mathcal{A}_\mu = (X, \{d_s\}_{s \in \Sigma}, Y, h)$  -also denoted as  $\Sigma^\mu$ -**tree automaton**- is a (non-initial)  $\Sigma$ -tree automaton with  $\Sigma = \Sigma^\mu$  comprising a unique ternary operation symbol  $s_\mu$  denoting the median operation  $\mu$  of a distributive lattice<sup>15</sup>  $\mathcal{X} = (X, \leq)$  and a set of nullary operation symbols corresponding to some of the terminal nodes of the labelled trees to be computed by the automaton, namely  $\Sigma^\mu = (\{s_\mu\} \cup S_0, \alpha)$  with  $\alpha : \{s_\mu\} \cup S_0 \rightarrow \mathbb{Z}_+$  such that  $\alpha(s_\mu) = 3$ ,  $\alpha(s) = 0$  for each  $s \in S_0$  and  $d_{s_\mu} = \mu$ . Given our present focus on a fixed population  $N$  of agents of size  $n = |N|$  we may conveniently take  $S_0$  such that  $|S_0| = 2 + 2^n$ : the elements of  $S_0$  correspond to 0, 1 (standing, respectively, for the bottom and top elements of the lattice) and to  $2^n$  ‘phantom votes’, one for each coalition. We can also posit  $Y = X$  and  $h = id$  i.e. we take states to be observable hence we can identify state and output spaces: it follows that in the ensuing analysis we may safely identify, with a slight abuse of language, the (non-initial)  $\Sigma^\mu$ -tree automaton  $\mathcal{A}_\mu$  and a  $\Sigma^\mu$ -algebra on  $X$ . An *initial*  $\Sigma^\mu$ -tree automaton  $\mathcal{A}_\mu^{I, \lambda}$  amounts to a  $\Sigma^\mu$ -tree automaton as supplemented with an initialization  $\lambda : I \rightarrow X$  i.e. an interpretation in  $X$  of variables in  $I$ : here, we take  $|I| = n$ , and the initialization  $\lambda$  models a particular ballot profile. Therefore,  $\mathcal{A}_\mu^{I, \lambda}$  embodies an interpretation in  $X$  of all terminal nodes of *any* finite labelled  $(\Sigma^\mu, I)$ -tree  $T$  and is ready to compute an output of  $T$  in  $X$  -the *behaviour* of  $\mathcal{A}_\mu$  at  $T$ -as given by the value  $\mathcal{A}_\mu^{I, \lambda}(T)$  of its run map at  $T$ . Thus, the behaviour of median  $\Sigma^\mu$ -tree automaton  $\mathcal{A}_\mu$  at any finite labelled  $(\Sigma^\mu, I)$ -tree  $T$  is the outcome of a nested sequence of medians  $\mu(\dots\mu(\mu(u, x_i, z), x_i, \mu(u', x_i, z'))\dots)$  starting with medians of

<sup>15</sup>Namely, a ternary operation on  $X$  satisfying Birkhoff-Kiss axioms m(i)-m(ii)-m(iii)-m(iv) (see Note 12 above).

projections  $x_i$  of  $x_N$ ,  $i = 1, \dots, n$  and the  $2^n$  elements of  $S_0 \subseteq X$  as dictated by  $T$  in the following manner. Terminal nodes of paths of maximum length  $l$  come by construction in  $2^{n-1}$  *triples* that share an immediate predecessor labelled by  $s_\mu$  i.e. the symbol of the (lattice) median operation. The nodes of any such triple are labelled by  $x_n$  and two distinct elements of  $S_0$ . For any  $k \leq l-1$ , the terminal nodes of paths of length  $l-k$  are labelled by  $x_{n-k}$  (the example below provides a very simple illustration with  $n = 3$ ).

**Example 10.** Let  $\mathcal{X} = (X, \leq)$  be a distributive lattice with bottom and top elements denoted by  $\perp$  and  $\top$ , respectively, and  $f : X^3 \rightarrow X$  the voting rule for  $(\{1, 2, 3\}, X)$  defined as follows: for any  $x = (x_1, x_2, x_3) \in X^3$ ,

$$f(x_N) = \mu(\mu(\mu(f(\perp, \perp, \perp), x_3, f(\perp, \perp, \top)), x_2, \mu(f(\perp, \top, \perp), x_3, f(\perp, \top, \top))), x_1, \mu(\mu(f(\top, \perp, \perp), x_3, f(\top, \perp, \top)), x_2, \mu(f(\top, \top, \perp), x_3, f(\top, \top, \top))))).$$

Then,  $f$  is  $l$ -median tree-automata representable: to see this, just consider for any  $x = (x_1, x_2, x_3) \in X^3$  the corresponding labelled tree

Thus, we say that a voting rule is representable by a (finitary)  $l$ -median tree automaton-or ***l*-median tree-automata representable (l-MTAR)** - if it can be regarded as the behaviour of some median  $\Sigma$ -tree automaton  $\mathcal{A}_\mu$  as properly initialized and acting on suitably labelled trees. That is made precise by the following:

**Definition 8. (*l*-median tree-automata representable (l-MTAR) voting rules)** Let  $\mathcal{X} = (X, \leq)$  be a bounded distributive lattice. Then, a voting rule  $f : X^N \rightarrow X$  is ***l*-median tree-automata representable (l-MTAR)** if there exists a  $\Sigma^\mu$ -tree automaton  $\mathcal{A}_\mu$  such that for any  $x_N \in X^N$  and any finite labelled  $(\Sigma^\mu, I)$ -tree  $T$  there is a corresponding initial  $\Sigma^\mu$ -tree automaton  $\mathcal{A}_\mu^{I, \lambda}$  with  $f(x_N) = \mathcal{A}_\mu^{I, \lambda}(T)$ .

We are now ready to state the main result of this paper concerning the characterization of strategy-proof voting rules on unimodal profiles. Our characterization result relies on the following three lemmas.

The first lemma simply establishes the equivalence between  $B_{\mathcal{X}}$ -monotonicity with respect to an arbitrary distributive lattice  $\mathcal{X}$  and *strategy-proofness* on the corresponding full unimodal domain  $U_{\mathcal{X}}^N$ .

**Lemma 1.** Let  $\mathcal{X} = (X, \leq)$  be a distributive lattice, and  $f : X^N \rightarrow X$  a voting rule for  $(N, X)$ . Then, the following statements are equivalent:

- (i)  $f$  is  $B_{\mathcal{X}}$ -monotonic;
- (ii)  $f$  is strategy-proof on  $U_{\mathcal{X}}^N$ ;
- (iii)  $f$  is strategy-proof on  $S_{\mathcal{X}}^N$ .

**Remark 3.** Lemma 1 above extends Lemma 1 of Danilov (1994) (concerning linear orders in a tree that are unimodal with respect to tree-betweenness). It also extends Proposition 3.2 of Nehring and Puppe (2007 (a)) (concerning locally strictly unimodal domains in a *finite* distributive -actually, Boolean-lattice) since it holds for *both* the (full) unimodal and the (full) locally strictly unimodal domain in *any* distributive lattice (including infinite and non-boolean ones). However, strictly speaking, Lemma 1 is not a generalization of Nehring and Puppe's result since the latter

concerns all social choice functions (not just voting rules), and any ‘rich’ locally strictly unimodal subdomain.

Observe that a restricted voting rule may be strategy-proof on its restricted unimodal domain while being not monotonic (i.e. the implications from (ii) or (iii) to (i) of the previous lemma do not hold in general for restricted voting rules).

To see this, consider the following example, adapted from Barberà, Berga and Moreno (2010), and slightly simplified: take  $X = \{a, b, c, d\}$  with  $a, b, c, d$  mutually distinct,  $\Delta_X = \{(x, x) : x \in X\}$ ,

$$\leq^* = \{(a, b), (a, c), (a, d), (b, c), (b, d), (d, c)\} \cup \Delta_X,$$

i.e.  $\mathcal{X}^* = (X, \leq^*)$  is the 4-chain.

$$\begin{aligned} \text{Then, posit} \quad \succ &= (a \succ b \succ c \sim d) \\ \succ' &= (d \succ' b \succ' c \sim' a) \\ \succ'' &= (a \succ'' b \succ'' c \succ'' d) \\ \succ''' &= (d \succ''' b \succ''' c \succ''' a) \end{aligned}$$

$D = \{\succ, \succ'\}$ ,  $D' = \{\succ'', \succ'''\}$ ,  $Y = \{a, d\}$  and define  $f' : Y^2 \times X^{N \setminus \{1,2\}} \rightarrow X$  by the following rule: for all  $x_{N \setminus \{1,2\}} \in X^{N \setminus \{1,2\}}$ ,

$$\begin{aligned} f'(a, a, x_{N \setminus \{1,2\}}) &= a, & f'(d, d, x_{N \setminus \{1,2\}}) &= d, \\ f'(a, d, x_{N \setminus \{1,2\}}) &= b, & f'(d, a, x_{N \setminus \{1,2\}}) &= c. \end{aligned}$$

First, observe that both  $\succ$  and  $\succ'$  are in  $U_{\mathcal{X}}^N$ , i.e. are unimodal, while  $\succ''$  and  $\succ'''$  are locally strictly unimodal: indeed,  $\text{top}(\succ) = \text{top}(\succ'') = a$ ,  $\text{top}(\succ') = \text{top}(\succ''') = d$  and it is immediately seen that:

$$B_{\mathcal{X}} = \left\{ \begin{array}{l} (a, b, c), (a, b, d), (a, d, c), (b, d, c), \\ (c, b, a), (d, b, a), (c, d, a), (c, d, b) \end{array} \right\} \cup \{(x, y, z) \in X^3 : x = y \text{ or } z = y\}.$$

But then, since  $\{(b, c), (b, d), (d, c)\} \cup \Delta_X$  is a subrelation of  $\succ$  and  $\{(b, c), (b, a), (d, a), (d, c)\} \cup \Delta_X$  is a subrelation of  $\succ'$ , it follows that unimodality of  $\succ$  and  $\succ'$  with respect to  $B_{\mathcal{X}}$  holds. Moreover,  $f'$  is by construction strategy-proof on  $D^2 \times U_{\mathcal{X}}^{N \setminus \{1,2\}}$  (and on  $(D')^2 \times S_{\mathcal{X}}^{N \setminus \{1,2\}}$ ): to check this, notice that 1 and 2 are the only non-dummy voters, and for all  $x_{N \setminus \{1,2\}} \in X^{N \setminus \{1,2\}}$ ,

$$\begin{aligned} f'(a, a, x_{N \setminus \{1,2\}}) &\succ f'(d, a, x_{N \setminus \{1,2\}}), & f'(a, d, x_{N \setminus \{1,2\}}) &\succ f'(d, d, x_{N \setminus \{1,2\}}), \\ f'(a, a, x_{N \setminus \{1,2\}}) &\succ f'(a, d, x_{N \setminus \{1,2\}}), & f'(d, a, x_{N \setminus \{1,2\}}) &\succ f'(d, d, x_{N \setminus \{1,2\}}), \end{aligned}$$

and similarly

$$\begin{aligned} f'(d, a, x_{N \setminus \{1,2\}}) &\succ' f'(a, a, x_{N \setminus \{1,2\}}), & f'(d, d, x_{N \setminus \{1,2\}}) &\succ' f'(a, d, x_{N \setminus \{1,2\}}), \\ f'(a, d, x_{N \setminus \{1,2\}}) &\succ' f'(a, a, x_{N \setminus \{1,2\}}), & f'(d, d, x_{N \setminus \{1,2\}}) &\succ' f'(d, a, x_{N \setminus \{1,2\}}), \end{aligned}$$

whence strategy-proofness of  $f'$  on  $D^2 \times U_{\mathcal{X}}^{N \setminus \{1,2\}}$  follows (strategy proofness on  $(D')^2 \times S_{\mathcal{X}}^{N \setminus \{1,2\}}$  follows from the same argument by replacing  $\succ''$  and  $\succ'''$  for  $\succ$  and  $\succ'$ , respectively).

However, observe that  $f'(d, a, x_{N \setminus \{1,2\}}) = c \notin [d, a] = [d, f'(a, a, x_{N \setminus \{1,2\}})]$  hence  $f'$  is *not*  $B_{\mathcal{X}}$ -monotonic.

The next lemma ensures that in an arbitrary distributive lattice the median operation as applied to voting rules does preserve  $B_{\mathcal{X}}$ -monotonicity.

**Lemma 2.** *Let  $\mathcal{X} = (X, \leq)$  be a distributive lattice, and  $f : X^N \rightarrow X$ ,  $g : X^N \rightarrow X$ ,  $h : X^N \rightarrow X$  voting rules that are  $B_{\mathcal{X}}$ -monotonic. Then  $\mu(f, g, h) : X^N \rightarrow X$  (where  $\mu(f, g, h)(x_N) = \mu(f(x_N), g(x_N), h(x_N))$  for all  $x_N \in X^N$ ) is also  $B_{\mathcal{X}}$ -monotonic.*

Finally, the next lemma - that only concerns *bounded* distributive lattices - provides a canonical median-based representation of all monotonic voting rules hence - in view of Lemma 1 above - of all strategy-proof voting rules on the corresponding full unimodal domain. That lemma relies on the notion of a *tree automaton* as defined above.

**Lemma 3.** *Let  $\mathcal{X} = (X, \leq)$  be a bounded distributive lattice and  $f : X^N \rightarrow X$  a  $B_{\mathcal{X}}$ -monotonic voting rule. Then,  $f$  is  $l$ -median tree-automata representable ( $l$ -MTAR).*

The main implications of the foregoing lemmas are indeed summarized by the following:

**Theorem 1.** *Let  $\mathcal{X} = (X, \leq)$  be a bounded distributive lattice,  $B_{\mathcal{X}}$  its latticial betweenness relation, and  $f : X^N \rightarrow X$  a voting rule for  $(N, X)$ . Then, the following statements are equivalent:*

- (i)  $f$  is  $B_{\mathcal{X}}$ -monotonic;
- (ii)  $f$  is strategy-proof on  $U_{\mathcal{X}}^N$ ;
- (iii)  $f$  is strategy-proof on  $S_{\mathcal{X}}^N$ ;
- (iv)  $f$  is  $l$ -MTAR;
- (v)  $f$  is a generalized weak committee voting rule.

**Remark 4.** Notice that Theorem 1 generalizes Moulin's characterization of strategy-proof voting rules on (full) unimodal domains in bounded chains to arbitrary bounded distributive lattices. Thus, it also offers a direct extension to all bounded distributive lattices of Moulin's original lattice-polynomial representation of strategy-proof voting rules to be contrasted with the alternative characterization via families of 'left-coalition systems' on (full) locally strictly unimodal domains in products of bounded chains due to Barberà, Gul and Stacchetti (1993), which relies heavily on the product-structure of the underlying lattices. In particular, Theorem 1 implies strategy-proofness of the simple majority voting rule on unimodal domains (with an odd population of voters), since it can be quite easily shown that the former is  $B_{\mathcal{X}}$ -monotonic (see e.g. Monjardet (1990) for a formal definition and study of the simple majority or extended median rule in a latticial framework). It follows that in an arbitrary bounded distributive lattice there exist voting rules - such as the simple majority rule - that jointly satisfy anonymity (i.e. symmetric treatment of voters), neutrality (i.e. symmetric treatment of outcomes), idempotence (i.e. faithful respect of unanimity of votes) and strategy-proofness on the full unimodal domain.

It can also be established, however, that strategy-proofness and coalitional strategy-proofness of a voting rule are *not* equivalent on unimodal domains in bounded distributive lattices. This is made precise by the following:

**Theorem 2.** *Let  $\mathcal{X} = (X, \leq)$  be a bounded distributive lattice. Then the following holds:*

- (i) *if  $|X| \geq 4$  then there exists a sublattice  $\mathcal{Y} = (Y, \leq_Y)$  of  $\mathcal{X}$  (with  $|Y| \geq 4$ ), subdomains  $D \subseteq U_{\mathcal{X}}$  and  $D' \subseteq S_{\mathcal{X}}$ , and a restricted voting rule  $f' : Y^2 \times X^{N \setminus \{1,2\}} \rightarrow X$  that is strategy-proof on  $D^2 \times U_{\mathcal{X}}^{N \setminus \{1,2\}}$  and on  $(D')^2 \times S_{\mathcal{X}}^{N \setminus \{1,2\}}$  but not coalitionally strategy-proof on  $D^2 \times U_{\mathcal{X}}^{N \setminus \{1,2\}}$  or on  $(D')^2 \times S_{\mathcal{X}}^{N \setminus \{1,2\}}$ ;*



(ii) if  $|X| \geq 4$  and  $\mathcal{X}$  is not a linear order then there exists a sublattice  $\mathcal{Y} = (Y, \leq_Y)$  of  $\mathcal{X}$  (with  $|Y| \geq 4$ ) and a voting rule  $f' : Y^N \rightarrow Y$  that is strategy-proof on  $U_Y^N$  and on  $S_Y^N$  but not coalitionally strategy-proof on  $U_Y^N$  or on  $S_Y^N$ .

Notice that if  $f : X^N \rightarrow X$  is strategy-proof on  $U_X^N$  and  $|X| \leq 3$  then  $f$  is also coalitionally strategy-proof on  $U_X^N$ : that implication follows from a straightforward adaptation of the proof of Theorem 1 of Barberà, Berga and Moreno (2010) to voting rules as combined with Proposition 1 of the same paper.

Moreover, as a further straightforward consequence of Theorem 2 (and of a few previously known results), we have the following:

**Corollary 1.** *Let  $\mathcal{X} = (X, \leq)$  be a bounded distributive lattice. Then the following statements are equivalent:*

- (i) *for each sublattice  $\mathcal{Y} = (Y, \leq_Y)$  of  $\mathcal{X}$  and each voting rule  $f : Y^N \rightarrow Y$ ,  $f$  is strategy-proof on  $U_Y^N$  (on  $S_Y^N$ , respectively) if and only if it is also coalitionally strategy-proof on  $U_Y^N$  (on  $S_Y^N$ , respectively);*
- (ii)  *$\mathcal{X} = (X, \leq)$  is a linear order.*

Thus, we have here a remarkable characterization of *bounded linear orders* as *the only bounded distributive lattices where equivalence of individual and coalitional strategy-proofness of voting rules on full unimodal domains holds*.

Indeed, the failure of equivalence between simple and coalitional strategy-proofness pointed out by Theorem 2 is readily extended to an impossibility result concerning availability of anonymous and idempotent coalitionally strategy-proof voting rules for full unimodal domains (and locally strictly unimodal domains) in a very general class of bounded distributive lattices, even if (full) neutrality is dropped. That is made precise by the following

**Theorem 3.** *Let  $\mathcal{X} = (X, \leq)$  be a bounded distributive lattice with at least two distinct atoms  $x, z \in X$  and  $\mathcal{Y} = (Y, \leq_Y)$  the sublattice of  $\mathcal{X}$  induced by the restriction of  $\leq$  to  $Y = \{0, x, z, x \vee z\}$ . Then, there is no anonymous voting rule  $f : X^N \rightarrow X$  which is locally sovereign and locally II-neutral on  $Y$ , and coalitionally strategy-proof on  $U_X^N$ , or on  $S_X^N$ .*

Thus, in sharp contrast to what happens in chains, no anonymous coalitionally strategy-proof voting rules are available on standard full unimodal or locally strictly unimodal domains *in bounded distributive lattices with at least two atoms*, including Boolean  $k$ -hypercubes with  $k > 1$ , even if an extended median-based aggregation rule is well-defined, and (weak) efficiency or even (full) sovereignty are *not* required at all.

#### 4. A SIMPLE EXAMPLE: SINGLE PEAKEDNESS AND STRATEGY-PROOFNESS IN THE BOOLEAN SQUARE

Consider a five-member committee  $N = \{1, 2, 3, 4, 5\}$  facing a decision problem concerning the formal requirements for candidates to fill a certain top corporate position. The committee is to decide whether (1) Mild -or Strict, i.e. more specific and demanding- formal qualifications and/or (2) Medium -or High- seniority are to be required of candidates. The outcome set is then  $\{(Mild, Medium), (Mild, High), (Strict, Medium), (Strict, High)\}$ : denoting both Mild and

Medium by 0 and both Strict and High by 1 the outcome set can be represented by the Boolean square  $\mathbf{2}^2 = (\{0, 1, x, y\}, \leq)$  where  $0 = (0, 0)$  denotes the bottom element,  $1 = (1, 1)$  denotes the top element, while  $x = (1, 0)$  and  $y = (0, 1)$  are not comparable. The latticial betweenness relation of  $\mathbf{2}^2$  as defined by the rule  $[(a, c, b) \in B_{\mathbf{2}^2} \text{ iff } a \wedge b \leq c \leq a \vee b]$  -where  $\wedge$  and  $\vee$  denote, respectively, the meet and join of  $\mathbf{2}^2$ - is

$$B_{\mathbf{2}^2} = \left\{ \begin{array}{l} (0, x, 1), (0, y, 1), (0, 0, 1), (0, 1, 1), (1, x, 0), (1, y, 0), (1, 0, 0), (1, 1, 0), \\ (x, 0, y), (x, 1, y), (x, x, y), (x, y, y), (y, 0, x), (y, 1, x), (y, x, x), (y, y, x), \\ (0, 0, x), (0, x, x), (x, 0, 0), (x, x, 0), (0, 0, y), (0, y, y), (y, 0, 0), (y, y, 0), \\ (x, x, 1), (x, 1, 1), (1, x, x), (1, 1, x), (y, y, 1), (y, 1, 1), (1, y, y), (1, 1, y) \end{array} \right\}.$$

An unimodal total preorder for  $\mathbf{2}^2$  is a total preorder  $\succsim$  on  $\{0, 1, x, y\}$  with a unique maximum that ‘respects’  $B_{\mathbf{2}^2}$ , namely such that for any  $(a, c, b) \in B_{\mathbf{2}^2}$  either  $c \succsim a$  or  $c \succsim b$  (or both). Then, if the complement of any element  $a$  is denoted  $a'$ , it is easily checked that the unimodal total preorders on  $\mathbf{2}^2$  are precisely *three* for each possible choice of the top outcome  $a \in \{0, x, y, 1\} = \{a, a', b, b'\}$  (hence *twelve* altogether), namely

$$\succsim_1 \equiv a \succ_1 b \succ_1 b' \sim_1 a', \quad \succsim_2 \equiv a \succ_2 b' \succ_2 b \sim_2 a', \quad \succsim_3 \equiv a \succ_3 b \sim_3 b' \sim_3 a'.$$

Indeed, the essential feature of an unimodal total preorder on  $B_{\mathbf{2}^2}$  is simply the following: it must be the case that *no second-best is both a complement of the first-best and strictly better than some other outcome*.

A voting rule  $f$  is  $B_{\mathbf{2}^2}$ -monotonic if -for any agent  $i$  and any profile  $z_{-i}$  of the other agents’ votes-  $i$ ’s vote for  $u$  ensures an outcome  $f(u, z_{-i})$  that lies between  $u$  and  $f(v, z_{-i})$  for any choice of  $v$  in  $\{0, 1, x, y\}$ .

The extended median or simple majority rule  $\mu^* : \{0, 1, x, y\}^5 \rightarrow \{0, 1, x, y\}$  is defined as follows: for any  $(a_1, a_2, a_3, a_4, a_5) \in \{0, x, y, 1\}^5$ ,

$$\mu^*(a_1, a_2, a_3, a_4, a_5) = \vee_{T \subseteq N, |T| \geq 3} (\wedge_{i \in T} a_i).$$

Since projections  $f_i(a_1, a_2, a_3, a_4, a_5) = a_i$ ,  $i = 1, \dots, 5$  are obviously  $B_{\mathbf{2}^2}$ -monotonic, and as shown below (see Lemma 2) the median preserves  $B_{\mathbf{2}^2}$ -monotonicity, it follows that the median  $\mu : \{0, 1, x, y\}^3 \rightarrow \{0, 1, x, y\}$  as defined by the rule  $\mu(a, b, c) = (a \wedge b) \vee (b \wedge c) \vee (a \wedge c)$  is also  $B_{\mathbf{2}^2}$ -monotonic. (It can also be easily checked that *on the Boolean square* the median is also efficient, since no outcome with zero votes can be selected by  $\mu$ : that property, however, fails in higher dimensional Boolean hypercubes).

Now, it turns out that both the join and the meet of any  $(a, b) \in \{0, x, y, 1\}^2$  (and therefore the join and the meet of any finite subset of  $\{0, x, y, 1\}$ ) are representable as the median (iterated median, respectively) of two projections and one constant, namely

$$\begin{aligned} a \vee b &= (a \wedge b) \vee (b \wedge 1) \vee (a \wedge 1) = \mu(a, b, 1) \text{ and} \\ a \wedge b &= (a \wedge b) \vee (b \wedge 0) \vee (a \wedge 0) = \mu(a, b, 0). \end{aligned}$$

Since constants (when regarded as constant functions) are obviously  $B_{\mathbf{2}^2}$ -monotonic, it also follows (from the  $B_{\mathbf{2}^2}$ -monotonicity preservation property of the median as mentioned above) that joins and meets of any finite subset of  $\{0, x, y, 1\}$  may be reduced to iterated medians of projections and constants, and are therefore also  $B_{\mathbf{2}^2}$ -monotonic. The same argument applies in particular to the extended median or simple majority rule  $\mu^*$  to conclude that  $\mu^*$  is  $B_{\mathbf{2}^2}$ -monotonic (with  $n = 5$  voters a direct check is of course still manageable, but the computation starts to become quite long and tedious).

Thus, since Theorem 1 implies that the strategy-proof voting rules for  $(N, \{0, x, y, 1\})$  on the full unimodal domain of total preorders in the Boolean square are precisely the  $B_{2^2}$ -monotonic functions on  $\{0, x, y, 1\}^N$ , it follows that  $\mu^*$  itself is in fact strategy-proof on that unimodal domain (along with all the  $B_{2^2}$ -monotonic functions on  $\{0, x, y, 1\}^5$ ).

Next, consider the following profile of total preorders:

$$\succsim_1 = \succsim_2 \equiv (x \succ 1 \succ 0 \sim y), \quad \succsim_3 = \succsim_4 \equiv (y \succ 1 \succ 0 \sim x), \quad \succsim_5 \equiv (0 \succ x \succ y \sim 1).$$

That profile is obviously unimodal with respect to  $B_{2^2}$  (see the definition above) since no second-best outcome is a complement of its first-best, and it is easily checked that

$$\mu^*(x, x, y, y, 0) = 0, \text{ whereas } \mu^*(1, 1, 1, 1, 0) = 1.$$

It follows that coalition  $\{1, 2, 3, 4\}$  can successfully manipulate  $\mu^*$  at that preference profile, namely the simple majority rule is *not coalitionally strategy-proof* on the full unimodal domain with respect to the latticial betweenness relation  $B_{2^2}$  (Theorem 2 shows that such a situation always occurs whenever the underlying bounded distributive lattice is not a chain). And all of the above can be generalized to an impossibility result: no anonymous voting rule enjoying a modicum of neutrality/sovereignty is coalitionally strategy-proof on unimodal domains in bounded distributive lattices that -like finite Boolean lattices (or hypercubes)  $2^m$ ,  $m \geq 2$ , and *unlike* chains- have at least two distinct atoms i.e. two elements that cover the bottom element (this is the content of Theorem 3).

To the the best of the authors' knowledge, none of those results on the Boolean *square* is available in the previous literature on strategy-proofness and single peakedness, and the same holds for counterparts of them under other notions of single peakedness. Partial results implying or suggesting some counterparts of Theorems 2 and 3 for finite Boolean lattices  $2^m$ ,  $m \geq 3$  under *alternative* notions of single peakedness are indeed available (including results on *separable preferences*, that are usually *not* presented as an instance of a single peaked domain but *can be*, as shown below): see e.g. Nehring and Puppe (2007(a),(b)), and Barberá, Sonnenschein and Zhou (1991). It should be stressed again, however, that such results -while interesting and valuable by themselves- are independent and at least in one respect narrower than those presented in the present work. That is so because the former not only ignore at all the unimodal case, but even in the locally strict unimodal case fail to cover -as opposed to Theorems 2 and 3 below- infinite lattices and the Boolean square (i.e. the finite Boolean case with  $m = 2$ ). Furthermore, it should be stressed that locally strict unimodality is somewhat at odds with a latticial outcome set. That is so because it relies on a notion of proximity-based betweenness-consistency that *cannot be backed by any metric consistent with standard latticial betweenness relations without allowing for multiple local peaks*.

In order to clarify those statements, it is worth reviewing here the sort of betweenness relations underlying such alternative notions of single peakedness.

The most widely used alternative version of single peakedness encountered in the extant literature requires that (i) there exist a unique maximum or best outcome, and (ii) any outcome  $x$  that lies between the best outcome and another outcome  $y$  distinct from  $x$  itself should also be *strictly better* than  $y$  (see e.g. Barberà, Gul and Stacchetti (1993), and Nehring and Puppe (2007 (a,b))). It is easily checked that under such notion of single peakedness (labeled '*general single peakedness*' in Nehring and Puppe (2007 (b))) and '*locally strict unimodality*' in the present paper) as applied to (the relevant part of) latticial betweenness, single peaked total preorders on  $2^2$  are -again- *three*

for each possible choice of the top outcome  $a \in \{0, x, y, 1\} = \{a, a', b, b'\}$  (hence *twelve* altogether), namely

$$\succsim'_1 \equiv a \succ'_1 b \succ_1 b' \succ'_1 a' , \quad \succsim'_2 \equiv a \succ'_2 b' \succ'_2 b \succ'_2 a' , \quad \succsim'_3 \equiv a \succ'_3 b \sim'_3 b' \succ'_3 a' .$$

Thus, as it is immediately checked, unimodal and locally strict unimodal total preorders comprise two *disjoint* sets.

Indeed, under locally strict unimodality, it is still possible to claim that every single voter's preferences are 'consistent' with the same betweenness relation, namely the latticial betweenness  $B_{2^2}$ . However, the implied 'consistency' is formulated in such a way that *only certain parts of  $B_{2^2}$  play an active role in shaping preferences*, and *distinct parts of it play such an active role for agents having distinct top outcomes*. In particular, and most remarkably, for each locally strictly unimodal total preorder  $\succsim$  on  $2^2$  with top outcome  $a$  both  $b \succ a'$  and  $b' \succ a'$  hold, while of course  $(b, a', b') \in B_{2^2}$ : therefore, when applied to  $B_{2^2}$  in its entirety, locally strictly unimodal total preorders actually admit *two* local peaks. That fact strongly suggests that *if* locally strictly unimodal total preorders are to be regarded as *single peaked* with respect to *some* betweenness relation on  $2^2$ , then claiming that role for the *entire* latticial betweenness  $B_{2^2}$ , while being of course a legitimate stipulation is *arguably somewhat far-fetched*. A far more natural and appropriate choice for that role would be apparently its proper subrelation  $B_{2^2}^a = B_{2^2} \setminus \{(b, a, b'), (b', a, b)\}$ . Observe, however, that such an approach would result in a 'non-classic' notion of single peakedness that makes reference to *several* preference-dependent hence, generally speaking, *agent-dependent betweenness relations (one for each possible best outcome)*.

To be sure, there is still another possibility to anchor all locally strictly unimodal total preorders to a *common betweenness relation*: that would entail choosing *another* proper subrelation of  $B_{2^2}$ , namely  $B_{2^2}^* = B_{2^2} \setminus \{(x, 1, y), (y, 1, x), (x, 0, y), (y, 0, x)\}$  as the relevant betweenness relation. Notice that  $B_{2^2}^*$  is in fact the natural betweenness relation of  $2^2$  when regarded not as a (Boolean) lattice, but *just as a partially ordered set*: thus, outcome  $b$  is declared to lie between  $a$  and  $c$  if and only if either  $a \leq b \leq c$  or  $c \leq b \leq a$  hold (the hallmark of order betweenness is that *no third outcome lies between two incomparable elements*). That move would enlarge the set of locally strictly unimodal total preorders, collapsing locally strict unimodality to uniqueness of the best outcome *whenever the best outcome is either  $x$  or  $y$* : in that case, we would end up with a notion of single peakedness for total preorders that relies on a unique shared betweenness relation but is itself *essentially preference-dependent* anyway. In any case, that choice of the relevant betweenness would be at variance with a full fledged treatment of outcome set  $2^2$  as a (distributive) *lattice*. It would result in a considerable relaxation of single peakedness restrictions, and a strengthening of the relevant notion of monotonicity, to the effect of rendering the median function *not* monotonic.<sup>16</sup>

*Separable preferences* on  $2^2$  (a notion due to Barberá, Sonnenschein and Zhou (1991) who define it for arbitrary finite Boolean lattices) are best introduced by regarding  $2^2$  as the power set of a two-item set  $\{x, y\}$  with  $x$  and  $y$  denoting singletons  $\{x\}$  and  $\{y\}$ , and the empty set  $\emptyset$  and  $\{x, y\}$  itself standing for 0 and 1, respectively. Any item  $a \in \{x, y\}$  is either 'good' or 'bad': it is *good* if  $\{a\} \succ \emptyset$  and *bad* if  $\emptyset \succ \{a\}$ . The set of all good items of a total preorder  $\succsim$  on  $2^2$  is denoted by  $G(\succsim)$ . A total preorder  $\succsim$  on  $2^2$  is *separable* if for any  $A \subseteq \{x, y\}$  and  $a \in \{x, y\} \setminus A$ ,  $A \cup \{a\} \succ A$  if and

<sup>16</sup>To see this, consider for instance  $(x, y, 1)$  and  $(y, y, 1)$ . Clearly,  $\mu(x, y, 1) = 1$ ,  $\mu(y, y, 1) = y$  and *not*  $B_{2^2}^*(x, 1, y)$  hence  $\mu$  is not  $B_{2^2}^*$ -monotonic, and no  $B_{2^2}^*$ -counterpart of Theorem 1 below applies to it.

only if  $a \in G(\succsim)$ . Clearly, for any separable total preorder  $\succsim$  on  $\mathbf{2}^2$ ,  $G(\succsim)$  is the unique maximum of  $\succsim$ . Resuming now for the sake of comparisons the standard notation used in the former discussion, the separable total preorders are three for each possible choice of the best outcome (that is, recall, the set of all good items), namely

$$\succsim_1'' \equiv a \succ_1'' b \succ_1'' b' \succ_1'' a' , \quad \succsim_2'' \equiv a \succ_2'' b' \succ_2'' b \succ_2'' a' , \quad \succsim_3'' \equiv a \succ_3'' b \sim_3'' b' \succ_3'' a'$$

where  $a$  denotes the set of all good items,  $a'$  is the complement of  $a$ , and  $b'$  is the complement of  $b$ .

Notice that on  $\mathbf{2}^2$  separable preferences are isomorphic to locally strictly unimodal preferences.<sup>17</sup> Thus, separable preferences are just locally strictly unimodal preferences on finite Boolean lattices in disguise. It follows that the same observations made on the latter also apply to separable preferences: precisely as locally strictly unimodal preferences, separable preferences can be regarded as single peaked either with respect to multiple, agent-dependent betweenness relations or with respect to a common betweenness relation. In both cases, however, *the betweenness relations involved and playing an active role depend on preferences, are distinct from latticial betweenness  $B_{\mathbf{2}^2}$  and, arguably, do disregard in relevant ways the latticial structure of the outcome set.*

## 5. RELATED LITERATURE AND CONCLUDING REMARKS

The main results of the present paper may be summarized as follows:

(i) Theorem 1 provides a characterization in terms of iterated medians of projections and constants of the class of strategy-proof voting rules on (full) unimodal domains and locally strictly unimodal domains *in all bounded distributive lattices*: thus, combining a version of the original Moulin's lattice-polynomial representation with a suitable generalization of ideas and techniques proposed by Danilov (1994) through an explicit reliance on tree automata, it extends in significant ways both Moulin (1980) and Danilov (1994) (which only concern *unimodal domains in bounded chains* and in *bounded trees*, respectively), and Barberà, Gul and Stacchetti (1993) and Nehring and Puppe (2007 (a),(b)) (which only concern *locally strictly unimodal domains in finite products of bounded chains* and in *finite distributive lattices*, respectively).

(ii) Theorem 2 establishes that *equivalence between (individual) strategy-proofness and coalitional strategy-proofness on both full unimodal and full locally strictly unimodal domains* holds precisely in bounded linear orders, and *fails in bounded distributive lattices that are not linear orders*: it complements the opposite results obtained by Moulin (1980) and Danilov (1994) for full unimodal domains in bounded chains and trees and by Barberà, Berga and Moreno (2010) for locally strictly unimodal domains in bounded chains, and extends previous results obtained by Barberà, Sonnenschein and Zhou (1991) and Nehring and Puppe (2007 (a),(b)) for locally strictly unimodal domains in certain *finite distributive lattices*.

(iii) Theorem 3 establishes -for bounded distributive lattices with at least two atoms- the impossibility of anonymous coalitional strategy-proof voting rules with even a minimal amount of local sovereignty and local neutrality on full unimodal domains: it also extends to a large subclass of non-sovereign voting rules for full unimodal and locally strictly unimodal domains in a much larger class of bounded distributive lattices some previous results mainly due to Barberà, Gul and Stacchetti (1993) and Nehring and Puppe (2007 (a), (b)) concerning voting rules for locally strictly unimodal

<sup>17</sup>The argument can be extended to arbitrary Boolean hypercubes  $\mathbf{2}^m$ ,  $m \geq 2$ .

domains in certain *finite* or *product* distributive lattices (indeed, a distributive lattice with two or more atoms need not be a product lattice, or finite).

In order to properly appreciate the significance of the foregoing result a few key contributions from the early literature on related issues are to be discussed in some detail.

The seminal paper by Moulin (see Moulin (1980)) provides an explicit characterization in terms of ‘extended medians’ of the class of all strategy-proof voting rules on the domain of *all* profiles of total preorders that are unimodal with respect to a fixed *bounded linear order*<sup>18</sup>. Furthermore, Moulin (1980) establishes the equivalence of strategy-proofness and coalitional strategy-proofness for *all* voting rules on such full unimodal domains. Clearly, Moulin’s result does not apply to the Boolean square. Moreover, its proof cannot be extended to the latter.

In fact, Moulin’s proof relies heavily on the following property of medians in bounded linear orders that *does not hold for medians in general bounded distributive lattices*: given an odd population of  $n = 2k + 1$  voters, for any  $(x_i)_{i=1, \dots, n} \in X^N$  the (extended) median  $\mu^*(x_1, \dots, x_n)$  i.e. the (iterated) median  $\mu(x_{2k}, \mu(x_{2(k-1)}, \mu(\dots(\mu(x_1, x_2, x_3))\dots), x_{2k-1}), x_{2k+1})$  is such that:

$$(5.1) \quad \min(|\{i \in N : x_i \leq \mu^*(x_1, \dots, x_n)\}|, |\{i \in N : \mu^*(x_1, \dots, x_n) \leq x_i\}|) \geq k + 1.$$

However, take  $n = 3$  (hence  $k + 1 = 2$ ) and consider the Boolean square  $\mathbf{2}^2$ .

Clearly  $\mu^*(1, x, y) = \mu(1, x, y) = 1$ , hence at  $(x_1, x_2, x_3) = (1, x, y)$ ,  $|\{i \in N : x_i \leq \mu^*(1, x, y)\}| = 3$ , but  $|\{i \in N : \mu^*(1, x, y) \leq x_i\}| = 1$ , and 5.1 fails.

In a similar vein, Danilov (1994) provides a characterization in terms of (iterated) medians of the class of strategy-proof voting rules on the domain of all unimodal *linear orders* (i.e. *antisymmetric* total preorders) when  $X$  is the vertex set of an undirected (bounded) *tree* (see also Danilov and Sotskov (2002) for further discussion of this topic, and Demange (1982) for an early study of majority-like voting rules on domains of unimodal linear orders in undirected trees).<sup>19</sup> Moreover, Danilov (1994) also shows that strategy-proofness and coalitional strategy-proofness of voting rules on *unimodal profiles of linear preference orders in undirected bounded trees* are *equivalent* properties. But in fact, it can be shown that Danilov’s proofs can be readily extended to the wider full domain of *unimodal total preference preorders* (arguing along the lines of the first part of the proof of Lemma 1 above), and to the case of an underlying *bounded linearly ordered set* of alternatives.

However, the key step of Danilov’s proof relies on the following property shared by *intervals* of linear orders and of undirected trees, namely:

$$(5.2) \quad \text{for all } x, y, v, z \in X \text{ such that } x \neq y, \text{ if } x \in [y, v] \text{ and } y \in [x, z] \text{ then } x \in [v, z].$$

<sup>18</sup>To be sure, Moulin proves the characterization result mentioned above for a *restricted* unimodal domain where voters are not allowed to regard the maximum or the minimum of the chain as their unique optimum. But Moulin’s proof can be adapted to the full unimodal domain.

<sup>19</sup>Danilov’s result can also be reformulated in terms of median tree-automata representable voting rules by suitably redefining the median operation in order to reflect the characteristic properties of medians of trees (as opposed to medians of bounded distributive lattices). This can be done replacing axiom  $m(i)$  of Birkhoff and Kiss (1947) (see note 12 above) with

$m'(i)$ : for all  $a, b, c, d$ ,  
either  $m(m(d, a, b), c, d) = m(d, a, c)$   
or  $m(m(d, a, b), c, d) = m(d, b, c)$   
(see Sholander (1952)).

Notice however that 5.2 does *not* hold for (lattice) intervals of arbitrary bounded distributive lattices. To see this, consider again precisely the Boolean square  $\mathbf{2}^2$  and notice that e.g.  $b \in [a, d]$ ,  $a \in [b, c]$  but  $b \notin [c, d]$ .<sup>20</sup>

Moulin’s fundamental characterization result also inspired many authors to explore related strategy-proofness issues on *other* single peaked domains (including locally strictly unimodal domains). Those alternative notions of single peakedness typically rely on binary proximity relations. Such proximity relations are usually jointly induced together *with* the betweenness relations *by* an underlying agent-invariant *metric* structure hence are agent-invariant and total (see e.g. Border and Jordan (1983), Barberà, Gul and Stacchetti (1993), Chichilnisky and Heal (1997), Ching (1997), Barberà, Massò and Neme (1997), Peremans, Peters, van der Stel and Storcken (1997), Schummer and Vohra (2002), Nehring and Puppe (2007(a), 2007(b)), Bordes, Laffond and Le Breton (2012), Chatterji, Sanver and Sen (2013)). Those works include several characterization results concerning some large subclass of strategy-proof voting mechanisms on suitably defined single-peaked domains (such as sovereign i.e. surjective voting rules or social choice functions). Moreover, some of those results provide valuable information on strategy-proofness properties of voting rules for certain proximity-based single peaked domains in certain distributive lattices and provide partial counterparts to our results on coalitional manipulability of median-based rules.

Thus, the existence of *sovereign* strategy-proof voting rules for Euclidean single peaked domains that are not coalitionally strategy-proof is well-established for proximity-based single peaked profiles in Euclidean  $m$ -dimensional spaces with  $m \geq 2$  (see e.g. Border and Jordan (1983), Peters, van der Stel and Storcken (1992), Peremans, Peters, van der Stel and Storcken (1997), Bordes, Laffond and Le Breton (2012)). Furthermore, Euclidean spaces are (unbounded) Riesz spaces i.e. are endowed with a natural (unbounded) distributive lattice structure. Notice however that the betweenness relation induced by the Euclidean metric is distinct from the lattice betweenness relation and in fact no well-behaved median operation is available in an  $m$ -dimensional Euclidean space with  $m \geq 2$ .

In their influential contribution, Barberà, Gul and Stacchetti (1993) identify single peakedness and locally strict unimodality and offer an alternative characterization of strategy-proof voting rules on (full) locally strictly unimodal domains in *(finite) products of bounded chains endowed with the  $L_1$ -metric*<sup>21</sup>: their characterization relies on *generalized median voter schemes* i.e. on representations of voting rules via families of outcome/dimension-specific generalized committees denoting winning coalitions. While that work mainly focusses on the finite case, it can be readily extended to finite products of arbitrary bounded chains. However, it relies heavily on the product structure of the underlying lattices.

Building upon some remarkable earlier contributions including Barberà, Gul and Stacchetti (1993) and Barberà, Massò and Neme (1997), and also focusing on *locally strict* unimodality (under the label ‘*generalized single peakedness*’), Nehring and Puppe (2007(a)) offer a comprehensive study and an

<sup>20</sup>See e.g. Sholander (1952, 1954(a)), Bandelt and Hedlíková (1983) for a thorough study of intervals in general median algebras, and Isbell (1980) for an even more general approach that also considers intervals in a larger class of ternary algebras.

<sup>21</sup>Namely,  $d^{L_1}(x, y) = \sum_i |x_i - y_i|$  for all  $x, y \in X$ . The  $L_1$ -metric is consistent with lattice betweenness in the sense explained in Remark 1 above. Barberà, Gul and Stacchetti (1993) also provides a characterization of strategy-proof social choice rules with range given by a product of sub-chains.

‘issue-by-issue voting-by-committees’-based characterization of sovereign (i.e. surjective) strategy-proof voting rules on rich domains of so-called locally strictly unimodal *linear orders* in certain *finite median interval spaces*<sup>22</sup> as induced by suitably defined ‘property spaces’. In particular, Nehring and Puppe (2007 (a), (b)) provide valuable results on locally strict unimodality and strategy-proofness in *finite* median spaces, and (finite) distributive lattices *are* a prominent instance of (finite) median spaces. However, due to their choice of *linear* preference domains as combined with their (locally) *strict* notion of unimodality, it turns out that their results are in fact irrelevant for the case of *unimodality* in finite distributive lattices other than finite chains, as explained in some detail in Section 2 above. Furthermore, Nehring and Puppe (2007 (b)) prove that the only efficient and strategy-proof voting rules on ‘rich’ domains of locally strictly unimodal profiles of linear orders in finite Boolean  $m$ -hypercubes with  $m \geq 3$  are weakly dictatorial. Notice, however, that efficient voting rules are in particular *sovereign* hence that result is not in any case an impossibility result for non-sovereign anonymous and coalitionally strategy-proof voting rules. It should also be stressed that the main result in Nehring, Puppe (2007 (b)) does entail equivalence-failure for simple and coalitional strategy-proofness in Boolean  $k$ -hypercubes for  $k \geq 3$  but it refers to a domain of locally strictly unimodal linear orders that is distinct -and in fact *disjoint* in any Boolean hypercube- from the domain of all unimodal total preorders which is the focus of the present work. Moreover, even on the locally strictly unimodal domain the foregoing result does not apply to the Boolean *square*, while our Theorem 2 covers the full unimodal domain in both infinite bounded distributive lattices and arbitrary Boolean hypercubes, including of course the Boolean square  $\mathbf{2}^2$ .

Another recent paper (Chatterji, Sanver and Sen (2013)) provides a characterization of those ‘*strongly-path-connected*’ domains of *linear orders* that ensure existence of anonymous and idempotent strategy-proof social choice functions for a voter population of *even* size: such characterization relies on a new, generalized notion of single-peakedness for linear orders denoted as ‘semi-single-peakedness’ (requiring essentially that the outcome set can be endowed with a tree structure such that locally strict unimodality as defined above holds within a certain threshold-distance from the top outcome). However, the domain of (all) unimodal *linear orders* on the Boolean square is -as observed above- empty hence it is trivially *not strongly path-connected*. Thus, the unimodal domain is definitely beyond the scope of the Chatterji-Sanver-Sen characterization.

Barberà, Berga and Moreno (2010) addresses the general issue of equivalence between simple and coalitional strategy-proofness and consider *locally strictly unimodal domains* of total preorders.<sup>23</sup> They establish that a property they newly introduce and label ‘*Sequential Inclusion*’ provides a general sufficient condition ensuring equivalence of individual and coalitional strategy-proofness, and show that *locally strictly unimodal domains of total preorders* as defined on a *linear order*  $(X, \leq)$  do satisfy it. Specifically, for each preference profile  $(\succsim_i)_{i \in N}$  *Sequential Inclusion* relies on a family of binary relations  $\succ (S((\succsim_i)_{i \in N}, y, z))$  as parameterized by ordered pairs  $(y, z)$  of outcomes and defined on  $S((\succsim_i)_{i \in N}, y, z)$ , the set of voters that strictly prefer  $y$  to  $z$  at  $(\succsim_i)_{i \in N}$ : in particular, voter pair  $(i, j)$  is in  $\succ (S((\succsim_i)_{i \in N}, y, z))$  if and only if  $i$  and  $j$  are in  $S((\succsim_i)_{i \in N}, y, z)$  and  $\{x \in X : z \succ_i x\} \subseteq \{x \in X : z \succ_j x\}$ . Of course any such  $\succ (S((\succsim_i)_{i \in N}, y, z))$  is reflexive:

<sup>22</sup>A median interval space amounts to a set with a ternary betweenness relation such that for any triple of elements there exists precisely one element which lies between each pair in the triple.

<sup>23</sup>It should be noticed, however, that locally strict unimodality in a bounded linear order reduces to unimodality when total preorders are in fact antisymmetric i.e. linear orders: details are available from the authors upon request.



Sequential Inclusion requires that all of them be also *connected and acyclic*. *Indirect Sequential Inclusion* is satisfied by a profile  $(\succsim_i)_{i \in N}$  if *either*  $(\succsim_i)_{i \in N}$  itself satisfy Sequential Inclusion *or* for each pair  $(y, z)$  of outcomes there exists a profile  $(\succsim'_i: i \in S((\succsim_i)_{i \in N}, y, z))$  such that: (i)  $y \succ'_i z$  for each  $i \in S((\succsim_i)_{i \in N}, y, z)$ , (ii)  $z \succ'_i x$  for each  $i \in S((\succsim_i)_{i \in N}, y, z)$  and each outcome  $x \neq z$  such that  $z \succsim_i x$ , and (iii)  $(S((\succsim'_i: i \in S((\succsim_i)_{i \in N}, y, z)))$  is connected and acyclic. A preference domain is then said to satisfy *Sequential Inclusion (Indirect Sequential Inclusion)* if each preference profile in that domain does satisfy it.<sup>24</sup>

Thus, the Barberà-Berga-Moreno result mentioned above extends to locally strictly unimodal domains Moulin's equivalence between individual and coalitional strategy-proofness on unimodal domains in bounded chains (see also Danilov and Sotskov (2002) and Le Breton and Zaporozhets (2009) in that connection). Notice, incidentally, that the Barberà-Berga-Moreno argument for such an equivalence result *cannot be extended to domains of unimodal total preorders even in bounded linear orders*.<sup>25</sup>

Finally, it should also be noticed that some of the results of the present paper - notably, Lemma 1 - can be easily reproduced in a more general setting e.g. in any median algebra (see Isbell (1980), Bandelt and Hedlíková (1983)). It remains to be seen which of the other results, if any, can also be lifted to the latter environment. This is however best left as a possible topic for future research.

## 6. APPENDIX 1: PROOFS

*Proof of Claim 1.* (i) If  $B_X(x, z, y)$  then  $x \wedge y \leq z \leq x \vee y$ . Since by definition  $x \wedge y = y \wedge x$  and  $x \vee y = y \vee x$  it obviously follows  $y \wedge x \leq z \leq y \vee x$  hence  $B_X(y, z, x)$  also holds.

(ii) Since by definition  $x \wedge y \leq x \leq x \vee y$  and  $x \wedge y \leq y \leq x \vee y$  hold for any  $x, y \in X$ , both  $B_X(x, x, y)$  and  $B_X(x, y, y)$  hold.

(iii) If  $B_X(x, y, x)$  then  $x = x \wedge x \leq y \leq x \vee x = x$  hence  $y = x$ .

(iv) If  $B_X(x, u, y)$ ,  $B_X(x, v, y)$  and  $B_X(u, z, v)$  then  $x \wedge y \leq u \leq x \vee y$ ,  $x \wedge y \leq v \leq x \vee y$  and  $u \wedge v \leq z \leq u \vee v$ .

Thus, by definition of  $\wedge$  and  $\vee$ ,  $x \wedge y \leq u \wedge v \leq x \vee y$  (that implies  $x \wedge y \leq z$ ) and  $x \wedge y \leq u \vee v \leq x \vee y$  (that implies  $z \leq x \vee y$ ). It follows that  $B_X(x, z, y)$  as required;

(v) If  $B_X(x, y, z)$  and  $B_X(y, x, z)$  then  $x \wedge z \leq y \leq x \vee z$  and  $y \wedge z \leq x \leq y \vee z$  hence

$$x = x \vee (y \wedge z) = (x \wedge (y \vee z)) \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) =$$

$$(x \wedge y) \vee (y \wedge z) \vee (x \wedge z) = (y \vee (x \wedge z)) \vee (x \wedge z) = y \vee (x \wedge z) = y.$$

□

*Proof of Claim 2.* First, notice that  $\succsim_x$  is, as suggested by notation, asymmetric: indeed, suppose not i.e.  $y \succsim_x z$  and  $z \succsim_x y$  for some  $y, z \in X$ . Then, in particular  $y \in [x, z]$  and  $z \in [x, y]$  hence  $y = z$  by antisymmetry of  $B_X$  as established above, a contradiction. Next, observe that  $\succsim_x$  is S-consistent: to check this, assume that on the contrary there exist  $y, z, z_1, \dots, z_k$  such that

<sup>24</sup>Moreover, it turns out that in our full unimodal setting Indirect Sequential Inclusion is a generalization of a similar 'richness' condition singled out by Le Breton and Zaporozhets (2009).

<sup>25</sup>Indeed, take a four-element *linear order*  $(\{x, y, w, z\}, \leq)$  such that  $x < y < z < w$ , consider total preorders  $\succsim_1, \succsim_2$  on  $X$  such that  $x \succsim_1 y \succsim_1 w \sim_1 z$  and  $z \succ_2 y \succ_2 w \sim_2 x$ : it can be quite easily shown that  $\succsim_1$  and  $\succsim_2$  are unimodal -though of course not strictly unimodal- and violate the 'Sequential Inclusion' property. Furthermore, it can be quite easily checked that the foregoing preference profile also fails to satisfy Indirect Sequential Inclusion (more details on all of the above are available from the authors upon request).

$y \in [x, z_1] \setminus \{z_1\}$ ,  $z_1 \in [x, z_2] \setminus \{z_2\}$ , ...,  $z_{k-1} \in [x, z_k] \setminus \{z_k\}$ ,  $z_k \in [x, z] \setminus \{z\}$  and  $z \in [x, y] \setminus \{y\}$ .

Then, in particular,

$$B_{\mathcal{X}}(x, y, z_1), B_{\mathcal{X}}(x, z_1, z_2), B_{\mathcal{X}}(x, z_2, z_3), \dots, B_{\mathcal{X}}(x, z_{k-1}, z_k), B_{\mathcal{X}}(x, z_k, z), B_{\mathcal{X}}(x, z, y).$$

It follows, by closure and convexity of  $B_{\mathcal{X}}$ ,

$$[x, y] \subseteq [x, z_1] \subseteq [x, z_2] \subseteq \dots \subseteq [x, z_k] \subseteq [x, z] \subseteq [x, y]$$

hence  $[x, y] = [x, z]$ .

Therefore,

$B_{\mathcal{X}}(x, z, y)$  and  $B_{\mathcal{X}}(x, y, z)$ , hence by symmetry  $B_{\mathcal{X}}(y, z, x)$  and  $B_{\mathcal{X}}(z, y, x)$  and by antisymmetry  $y = z$ , a contradiction since  $z \in [x, y] \setminus \{y\}$ : hence, S-consistency of  $\succ_x$  is established.

It follows that Suzumura's Theorem applies and  $\succ_x$  admits a non-trivial extension  $\succ_x^*$  that is a total preorder. Moreover, since  $\succ_x$  is asymmetric,  $\succ_x \subseteq \succ_x^*$  hence by construction  $\succ_x^*$  is locally strictly unimodal as required.  $\square$

*Proof of Lemma 1.* (i)  $\Rightarrow$  (ii) Let  $f$  be  $B_{\mathcal{X}}$ -monotonic with respect to  $\mathcal{X}$ . Now, consider any  $\succ = (\succ_j)_{j \in N} \in U_{\mathcal{X}}^N$  and any  $i \in N$ . By definition of  $B_{\mathcal{X}}$ -monotonicity  $f(\text{top}(\succ_i), x_{N \setminus \{i\}}) \in [\text{top}(\succ_i), f(x_i, x_{N \setminus \{i\}})]$  for all  $x_{N \setminus \{i\}} \in X^{N \setminus \{i\}}$  and  $x_i \in X$ . But then, since clearly

$\text{top}(\succ_i) \succ_i f(\text{top}(\succ_i), x_{N \setminus \{i\}})$ , either  $f(\text{top}(\succ_i), x_{N \setminus \{i\}}) = \text{top}(\succ_i)$  or  $f(\text{top}(\succ_i), x_{N \setminus \{i\}}) \succ_i f(x_i, x_{N \setminus \{i\}})$  by unimodality of  $\succ_i$ . Hence,  $f(\text{top}(\succ_i), x_{N \setminus \{i\}}) \succ_i f(x_i, x_{N \setminus \{i\}})$  in any case. It follows that  $f$  is indeed strategy-proof on  $U_{\mathcal{X}}^N$ .

(ii)  $\Rightarrow$  (i) Let us assume that  $f : X^N \rightarrow X$  is *not*  $B_{\mathcal{X}}$ -monotonic: thus, there exist  $i \in N$ ,  $x'_i \in X$  and  $x_N = (x_i)_{i \in N} \in X^N$  such that  $f(x_N) \notin [x_i, f(x'_i, x_{N \setminus \{i\}})]$ . Then, consider the total preorder  $\succ^*$  on  $X$  defined as follows:  $x_i = \text{top}(\succ^*)$  and for all  $y, z \in X \setminus \{x_i\}$ ,  $y \succ^* z$  if and only if (i)  $\{y, z\} \subseteq [x_i, f(x'_i, x_{N \setminus \{i\}})] \setminus \{x_i\}$  or (ii)  $y \in [x_i, f(x'_i, x_{N \setminus \{i\}})] \setminus \{x_i\}$  and  $z \notin [x_i, f(x'_i, x_{N \setminus \{i\}})]$  or (iii)  $y \notin [x_i, f(x'_i, x_{N \setminus \{i\}})]$  and  $z \in [x_i, f(x'_i, x_{N \setminus \{i\}})]$ . Clearly, by construction  $\succ^*$  consists of three indifference classes with  $\{x_i\}$ ,  $[x_i, f(x'_i, x_{N \setminus \{i\}})] \setminus \{x_i\}$  and  $X \setminus [x_i, f(x'_i, x_{N \setminus \{i\}})]$  as top, medium and bottom indifference classes, respectively. Now, observe that  $\succ^* \in U_{\mathcal{X}}$ . To check this statement, take any  $y, z, v \in X$  such that  $y \neq z$  and  $v \in [y, z]$  (if  $y = z$  then  $v = y = z$  and there is in fact nothing to prove). If  $\{y, z\} \subseteq [x_i, f(x'_i, x_{N \setminus \{i\}})]$  then  $B_{\mathcal{X}}(x_i, v, f(x'_i, x_{N \setminus \{i\}}))$  by convexity of  $B_{\mathcal{X}}$ , i.e.  $v \in [x_i, f(x'_i, x_{N \setminus \{i\}})]$ .  $v$  by construction  $x_i \wedge f(x'_i, x_{N \setminus \{i\}}) \leq y \wedge z \leq v \leq y \vee z \leq x_i \vee f(x'_i, x_{N \setminus \{i\}})$ , i.e.  $v \in [x_i, f(x'_i, x_{N \setminus \{i\}})]$ . Assume without loss of generality that  $y \neq x_i$ : it follows that  $v \succ^* y$  by definition of  $\succ^*$ . If on the contrary  $\{y, z\} \cap (X \setminus [x_i, f(x'_i, x_{N \setminus \{i\}})]) \neq \emptyset$  then clearly by definition of  $\succ^*$  there exists  $w \in [y, z]$  such that  $v \succ^* w$ . Thus,  $\succ^* \in U_{\mathcal{X}}$  as claimed. Also, by assumption  $f(x_N) \in X \setminus [x_i, f(x'_i, x_{N \setminus \{i\}})]$  whence by construction  $f(x'_i, x_{N \setminus \{i\}}) \succ^* f(x_N)$ . But then,  $f$  is *not* strategy-proof on  $U_{\mathcal{X}}^N$ .

(i)  $\Rightarrow$  (iii) Again, let  $f$  be  $B_{\mathcal{X}}$ -monotonic with respect to  $\mathcal{X}$ . Now, consider any  $\succ = (\succ_j)_{j \in N} \in S_{\mathcal{X}}^N$  and any  $i \in N$ . By definition of  $B_{\mathcal{X}}$ -monotonicity  $f(\text{top}(\succ_i), x_{N \setminus \{i\}}) \in [\text{top}(\succ_i), f(x_i, x_{N \setminus \{i\}})]$  for all  $x_{N \setminus \{i\}} \in X^{N \setminus \{i\}}$  and  $x_i \in X$ . But then, either  $f(\text{top}(\succ_i), x_{N \setminus \{i\}}) = f(x_i, x_{N \setminus \{i\}})$  or  $f(\text{top}(\succ_i), x_{N \setminus \{i\}}) \succ_i f(x_i, x_{N \setminus \{i\}})$  by locally strict unimodality of  $\succ_i$ . Hence,  $f(\text{top}(\succ_i), x_{N \setminus \{i\}}) \succ_i f(x_i, x_{N \setminus \{i\}})$  in any case. It follows that  $f$  is indeed strategy-proof on  $S_{\mathcal{X}}^N$ .

(iii)  $\Rightarrow$  (i) Let us assume that  $f : X^N \rightarrow X$  is *not*  $B_{\mathcal{X}}$ -monotonic: thus, there exist  $i \in N$ ,  $x'_i \in X$  and  $x_N = (x_i)_{i \in N} \in X^N$  such that  $f(x_N) \notin [x_i, f(x'_i, x_{N \setminus \{i\}})]$ . Then, consider a binary relation

$\succ'$  on  $X$  defined by the following clauses ( $\alpha$ )  $x_i = \text{top}(\succ')$  i.e.  $x_i \succ' y$  for all  $y \in X \setminus \{x_i\}$ ; ( $\beta$ ) if  $\{y, z\} \subseteq [x_i, f(x'_i, x_{N \setminus \{i\}})] \setminus \{x_i\}$  then  $y \succ' z$  if and only if  $z \in [x_i, y] \setminus \{y\}$ ; ( $\gamma$ ) if  $y \in [x_i, f(x'_i, x_{N \setminus \{i\}})] \setminus \{x_i\}$  and  $z \notin [x_i, f(x'_i, x_{N \setminus \{i\}})]$  then  $y \succ' z$ ; ( $\delta$ ) if  $\{y, z\} \subseteq X \setminus [x_i, f(x'_i, x_{N \setminus \{i\}})]$  then  $y \succ' z$  if and only if  $z \in [x_i, y] \setminus \{y\}$ .

Then, observe that  $\succ' \cap \{[x_i, f(x'_i, x_{N \setminus \{i\}})]\}^2$  and  $\succ' \cap \{X \setminus [x_i, f(x'_i, x_{N \setminus \{i\}})]\}^2$  amount to the restrictions of  $\succ_{x_i}$  as defined above (see Claim 2) to  $[x_i, f(x'_i, x_{N \setminus \{i\}})]$  and  $X \setminus [x_i, f(x'_i, x_{N \setminus \{i\}})]$ , respectively, hence by Claim 2 are S-consistent and therefore admits non-trivial extensions to total preorders  $\succ'_1$  and  $\succ'_2$  on their respective *disjoint* domains. It follows that  $\succ' = \succ'_1 \oplus \succ'_2$  as defined by the rule  $x \succ' y$  if and only if  $(x \succ'_1 y, x \succ'_2 y \text{ or } x \in [x_i, f(x'_i, x_{N \setminus \{i\}})] \text{ and } y \in X \setminus [x_i, f(x'_i, x_{N \setminus \{i\}})])$  is a total preorder on  $X$ . Moreover,  $\succ'$  is locally strictly unimodal by construction i.e.  $\succ' \in S_{\mathcal{X}}^N$ . Also, by assumption  $f(x_N) \in X \setminus [x_i, f(x'_i, x_{N \setminus \{i\}})]$  whence by construction  $f(x'_i, x_{N \setminus \{i\}}) \succ' f(x_N)$ . But then,  $f$  is *not* strategy-proof on  $S_{\mathcal{X}}^N$ .  $\square$

*Proof of Lemma 2.* Take any  $x_N \in X^N$ . By definition of  $B_{\mathcal{X}}$ -monotonicity, it suffices to show that for any  $i \in N$  and  $x'_i \in X$ ,  $\mu(f, g, h)(x_N) \in [x_i, \mu(f, g, h)(x'_i, x_{N \setminus \{i\}})]$ . Indeed, by monotonicity of  $f, g, h$  with respect to  $\mathcal{X}$ ,  $f(x_N) \in [x_i, f(x'_i, x_{N \setminus \{i\}})]$ ,  $g(x_N) \in [x_i, g(x'_i, x_{N \setminus \{i\}})]$ , and  $h(x_N) \in [x_i, h(x'_i, x_{N \setminus \{i\}})]$ .

A change of variables is in order here for the sake of convenience, namely  $x_f = f(x_N)$ ,  $x'_f = f(x'_i, x_{N \setminus \{i\}})$ ,  $x_g = g(x_N)$ ,  $x'_g = g(x'_i, x_{N \setminus \{i\}})$ ,  $x_h = h(x_N)$ ,  $x'_h = h(x'_i, x_{N \setminus \{i\}})$ , whence  $\mu(f, g, h)(x_N) = \mu(x_f, x_g, x_h)$ , and  $\mu(f, g, h)(x'_i, x_{N \setminus \{i\}}) = \mu(x'_f, x'_g, x'_h)$ . Thus,  $x_i \wedge x'_l \leq x_l \leq x_i \vee x'_l$ ,  $l = f, g, h$ , by hypothesis, while the thesis amounts to  $x_i \wedge \mu(x'_f, x'_g, x'_h) \leq \mu(x_f, x_g, x_h) \leq x_i \vee \mu(x'_f, x'_g, x'_h)$ . Now,  $\mu(x'_f, x'_g, x'_h) = (x'_f \wedge x'_g) \vee (x'_g \wedge x'_h) \vee (x'_f \wedge x'_h)$  hence by distributivity and the basic latticial identities we get:

$$\begin{aligned} & x_i \wedge ((x'_f \wedge x'_g) \vee (x'_g \wedge x'_h) \vee (x'_f \wedge x'_h)) \\ = & (x_i \wedge (x'_f \wedge x'_g)) \vee (x_i \wedge (x'_g \wedge x'_h)) \vee (x_i \wedge (x'_f \wedge x'_h)) \\ = & ((x_i \wedge x'_f) \wedge (x_i \wedge x'_g)) \vee ((x_i \wedge x'_g) \wedge (x_i \wedge x'_h)) \vee ((x_i \wedge x'_f) \wedge (x_i \wedge x'_h)). \end{aligned}$$

However, by hypothesis, distributivity and the basic latticial identities again:

$$\begin{aligned} & ((x_i \wedge x'_f) \wedge (x_i \wedge x'_g)) \vee ((x_i \wedge x'_g) \wedge (x_i \wedge x'_h)) \vee ((x_i \wedge x'_f) \wedge (x_i \wedge x'_h)) \\ \leq & (x_f \wedge x_g) \vee (x_g \wedge x_h) \vee (x_f \wedge x_h) = \mu(x_f, x_g, x_h) \\ \leq & ((x_i \vee x'_f) \wedge (x_i \vee x'_g)) \vee ((x_i \vee x'_g) \wedge (x_i \vee x'_h)) \vee ((x_i \vee x'_f) \wedge (x_i \vee x'_h)) \\ = & (x_i \vee (x'_f \wedge x'_g)) \vee (x_i \vee (x'_g \wedge x'_h)) \vee (x_i \vee (x'_f \wedge x'_h)) \\ = & x_i \vee ((x'_f \wedge x'_g) \vee (x'_g \wedge x'_h) \vee (x'_f \wedge x'_h)) = x_i \vee \mu(x'_f, x'_g, x'_h) \end{aligned}$$

as required.  $\square$

*Proof of Lemma 3.* Take any  $x_{N \setminus \{1\}} \in X^{N \setminus \{1\}}$  and consider  $f_{x_{N \setminus \{1\}}} : X \rightarrow X$  as defined by the rule  $f_{x_{N \setminus \{1\}}}(x_1) = f(x_1, x_{N \setminus \{1\}})$  for all  $x_1 \in X$ . Thus, by definition  $f_{x_{N \setminus \{1\}}}$  is  $B_{\mathcal{X}}$ -monotonic with respect to  $(X, \leq)$ , i.e.  $f_{x_{N \setminus \{1\}}}(x) \in [x, f_{x_{N \setminus \{1\}}}(y)]$ , namely

$$x \wedge f_{x_{N \setminus \{1\}}}(y) \leq f_{x_{N \setminus \{1\}}}(x) \leq x \vee f_{x_{N \setminus \{1\}}}(y) \text{ for any } x, y \in X.$$

In particular,

$$\begin{aligned}
\perp &= \perp \wedge f_{x_N \setminus \{1\}}(x_1) \leq f_{x_N \setminus \{1\}}(\perp) \leq \perp \vee f_{x_N \setminus \{1\}}(x_1) = f_{x_N \setminus \{1\}}(x_1), \\
f_{x_N \setminus \{1\}}(x_1) &= \top \wedge f_{x_N \setminus \{1\}}(x_1) \leq f_{x_N \setminus \{1\}}(\top) \leq \top \vee f_{x_N \setminus \{1\}}(x_1) = \top, \\
x_1 \wedge f_{x_N \setminus \{1\}}(\perp) &\leq f_{x_N \setminus \{1\}}(x_1) \leq x_1 \vee f_{x_N \setminus \{1\}}(\perp), \text{ and} \\
x_1 \wedge f_{x_N \setminus \{1\}}(\top) &\leq f_{x_N \setminus \{1\}}(x_1) \leq x_1 \vee f_{x_N \setminus \{1\}}(\top), \text{ for all } x_1 \in X.
\end{aligned}$$

Now, take any  $x_1 \in X$  and consider  $\mu(f_{x_N \setminus \{1\}}(\perp), x_1, f_{x_N \setminus \{1\}}(\top))$ . By definition, and distributivity of  $(X, \leq)$ ,

$$\begin{aligned}
&\mu(f_{x_N \setminus \{1\}}(\perp), x_1, f_{x_N \setminus \{1\}}(\top)) \\
&= (x_1 \wedge f_{x_N \setminus \{1\}}(\perp)) \vee (x_1 \wedge f_{x_N \setminus \{1\}}(\top)) \vee (f_{x_N \setminus \{1\}}(\perp) \wedge f_{x_N \setminus \{1\}}(\top)) \\
&= (x_1 \vee f_{x_N \setminus \{1\}}(\perp)) \wedge (x_1 \vee f_{x_N \setminus \{1\}}(\top)) \wedge (f_{x_N \setminus \{1\}}(\perp) \vee f_{x_N \setminus \{1\}}(\top)).
\end{aligned}$$

Since by  $B_{\mathcal{X}}$ -monotonicity, as observed above,

$$\begin{aligned}
x_1 \wedge f_{x_N \setminus \{1\}}(\perp) &\leq f_{x_N \setminus \{1\}}(x_1), \quad x_1 \wedge f_{x_N \setminus \{1\}}(\top) \leq f_{x_N \setminus \{1\}}(x_1), \text{ and} \\
f_{x_N \setminus \{1\}}(\perp) \wedge f_{x_N \setminus \{1\}}(\top) &= f_{x_N \setminus \{1\}}(\perp) \leq f_{x_N \setminus \{1\}}(x_1),
\end{aligned}$$

it follows that  $\mu(f_{x_N \setminus \{1\}}(\perp), x_1, f_{x_N \setminus \{1\}}(\top)) \leq f_{x_N \setminus \{1\}}(x_1)$ . Similarly,

$$\begin{aligned}
f_{x_N \setminus \{1\}}(x_1) &\leq x_1 \vee f_{x_N \setminus \{1\}}(\perp), \quad f_{x_N \setminus \{1\}}(x_1) \leq x_1 \vee f_{x_N \setminus \{1\}}(\top), \text{ and} \\
f_{x_N \setminus \{1\}}(x_1) &\leq f_{x_N \setminus \{1\}}(\top) = f_{x_N \setminus \{1\}}(\perp) \vee f_{x_N \setminus \{1\}}(\top).
\end{aligned}$$

It follows that  $f_{x_N \setminus \{1\}}(x_1) \leq \mu(f_{x_N \setminus \{1\}}(\perp), x_1, f_{x_N \setminus \{1\}}(\top))$  as well, whence

$$\begin{aligned}
f_{x_N \setminus \{1\}}(x_1) &= \mu(f_{x_N \setminus \{1\}}(\perp), x_1, f_{x_N \setminus \{1\}}(\top)) = \mu(f_{x_N \setminus \{1\}}(\perp), \pi_1(x_1), f_{x_N \setminus \{1\}}(\top)) \\
&= \mu(f(\perp, x_N \setminus \{1\}), \pi_1(x_1), f(\top, x_N \setminus \{1\})), \text{ i.e.} \\
f_{x_N \setminus \{1\}} &= \mu(f(\perp, x_N \setminus \{1\}), \pi_1, f(\top, x_N \setminus \{1\})).
\end{aligned}$$

Thus, for all  $x_1 \in X$ ,  $f_{x_N \setminus \{1\}}(x_1)$  is the *first* term of the nested sequence of medians that provides the run of the median tree-automaton  $\mathcal{A}_\mu^{I, \lambda}$  as initialized with ballot profile  $x_N$  and applied to the finite  $(\Sigma^\mu, I)$ -tree  $T = T(x_N, \{f(x^*)\}_{x^* \in \{\perp, \top\}^N})$  with terminal nodes suitably labelled by projections of  $x_N$  and elements of  $\{f(x^*)\}_{x^* \in \{\perp, \top\}^N}$ .

Next, consider  $f_{x_N \setminus \{1, 2\}} : X^2 \rightarrow X$  as defined by the following rule: for all  $x_1, x_2 \in X$ ,

$$\begin{aligned}
f_{x_N \setminus \{1, 2\}}(x_1, x_2) &= f(x_1, x_2, x_N \setminus \{1, 2\}) = \mu(f(\perp, x_N \setminus \{1\}), \pi_1(x_1), f(\top, x_N \setminus \{1\})) \\
&= \mu(f(\perp, x_2, x_N \setminus \{1, 2\}), \pi_1(x_1), f(\top, x_2, x_N \setminus \{1, 2\})).
\end{aligned}$$

By repeating the previous argument of this proof as applied to both  $f(\perp, x_2, x_N \setminus \{1, 2\})$  and  $f(\top, x_2, x_N \setminus \{1, 2\})$ , it follows that:

$$\begin{aligned}
f_{x_N \setminus \{1, 2\}}(x_1, x_2) &= \mu(\mu(f(\perp, \perp, x_N \setminus \{1, 2\}), \pi_2(x_2), f(\perp, \top, x_N \setminus \{1, 2\})), \\
&\quad \pi_1(x_1), \mu(f(\top, \perp, x_N \setminus \{1, 2\}), \pi_2(x_2), f(\top, \top, x_N \setminus \{1, 2\}))),
\end{aligned}$$

i.e.  $f_{x_N \setminus \{1, 2\}}$  is the *fourth* term of the nested sequence of medians that provides the run of the median tree-automaton  $\mathcal{A}_\mu^{I, \lambda}$  as initialized with ballot profile  $x_N$  and applied to the finite  $(\Sigma^\mu, I)$ -tree  $T = T(x_N, \{f(x^*)\}_{x^* \in \{\perp, \top\}^N})$  as mentioned above. Repeated iteration of the very same argument

establishes that, for all  $x_N \in X^N$ ,  $f(x_N) = \mathcal{A}_\mu^{I,\lambda}(T)$  i.e.  $f(x_N)$  is the value of the behaviour of  $\mathcal{A}_\mu^{I,\lambda}$  at  $T$ .  $\square$

*Proof of Theorem 1.* (i)  $\iff$  (ii) It follows from Lemma 1.

(i)  $\iff$  (iii) It also follows from Lemma 1.

(i)  $\implies$  (iv) Immediate from Lemma 3.

(iv)  $\implies$  (i) It follows immediately from the definition of l-median TA-representation, from the observation that projections and constants induce  $B_{\mathcal{X}}$ -monotonic voting rules, and from Lemma 2.

(iv)  $\implies$  (v) Suppose  $f$  has a  $\Sigma^\mu$ -tree automata representation. Then, by construction, there exists a set of at most  $2^{|N|}$  distinct elements namely

$f[\{\perp, \top\}^N] = \left\{ f(z_N) : z_N \in \{\perp, \top\}^N \right\}$  such that  $f(x_N)$  is the output of an l-median tree automaton with input set  $I(f(x_N)) := f[\{\perp, \top\}^N] \cup \{x_i : i \in N\}$ . But then, by definition of the output computation rule of an l-median tree automaton it is immediately checked that  $f(x_N)$  is the join of a finite set of meets  $\bigwedge_{j=1}^k z_j$ ,  $k \leq 2^{|N|}$ ,  $z_i \in I(f(x_N))$ ,  $i = 1, \dots, k$ . Hence, by positing

$$y_S[\bigwedge_{j=1}^k z_j] = \bigwedge \{z : z = z_j \text{ for some } j = 1, \dots, k \text{ and } z_j \neq x_i \text{ for all } i \in N\}$$

where  $S[\bigwedge_{j=1}^k z_j] := \{i \in N : x_i = z_h \text{ for some } h = 1, \dots, k\}$ , and

$$y_S^* = \bigvee \{y_S[\bigwedge_{j=1}^k z_j] \text{ such that } S[\bigwedge_{j=1}^k z_j] = S\},$$

and noticing that  $y_S$  thus defined does not depend on  $x_N$ ,

it follows that there exists a nonempty  $\mathcal{C} \subseteq \mathcal{P}(N)$  such that

$$f(x_N) = \bigvee_{S \in \mathcal{C}} ((\bigwedge_{i \in S} x_i) \wedge y_S^*).$$

Moreover, by putting  $y_T^* = \perp$  for any  $T \in \mathcal{P}(N) \setminus \mathcal{C}$ , it also follows that

$$f(x_N) = \bigvee_{S \subseteq N} ((\bigwedge_{i \in S} x_i) \wedge y_S^*)$$

hence  $f$  is indeed a generalized weak committee voting rule.

(v)  $\implies$  (i) Let  $f : X^N \rightarrow X$  be a generalized weak committee voting rule i.e. there exists an order filter  $\mathcal{F}$  of  $(\mathcal{P}(N), \subseteq)$  such that  $f(x_N) = \bigvee_{S \in \mathcal{F}} ((\bigwedge_{i \in S} x_i) \wedge y_S^*)$  for all  $x_N \in X^N$ . Then, observe that -for any  $x, y \in X$ -  $x \wedge y = \mu(x, y, \perp)$  and  $x \vee y = \mu(x, y, \top)$ . Hence by repeated application of Lemma 2 it follows that  $f$  is  $B_{\mathcal{X}}$ -monotonic.  $\square$

*Proof of Theorem 2.* (i) Take restricted voting rule  $f'$  as introduced in Remark 1 above, where it was also shown that  $f'$  is strategy-proof on  $D^2 \times U_{\mathcal{X}}^{N \setminus \{1,2\}}$  and on  $(D')^2 \times S_{\mathcal{X}}^{N \setminus \{1,2\}}$ .

Now, to address the strategy-proofness issue on  $D^2 \times U_{\mathcal{X}}^{N \setminus \{1,2\}}$  consider any preference profile  $(\succsim_i)_{i \in N}$  such that  $\succsim_1 = \succsim'$  and  $\succsim_2 = \succsim$  hence  $\text{top}(\succsim_1) = d$ ,  $\text{top}(\succsim_2) = a$ . Then, for any  $x_{N \setminus \{1,2\}} \in X^{N \setminus \{1,2\}}$ , both

$$\begin{aligned} f'(a, d, x_{N \setminus \{1,2\}}) &\succ_1 f'(\text{top}(\succsim_1), \text{top}(\succsim_2), x_{N \setminus \{1,2\}}) \text{ and} \\ f'(a, d, x_{N \setminus \{1,2\}}) &\succ_2 f'(\text{top}(\succsim_1), \text{top}(\succsim_2), x_{N \setminus \{1,2\}}), \end{aligned}$$

it follows that coalition  $\{1, 2\}$  can manipulate the outcome at  $(\succsim_i)_{i \in N}$  namely  $f'$  is *not* coalitionally strategy-proof on  $D^2 \times U_{\mathcal{X}}^{N \setminus \{1,2\}}$ . Strategy-proofness (and failure of coalitional strategy-proofness) of  $f'$  on  $(D')^2 \times S_{\mathcal{X}}^{N \setminus \{1,2\}}$  is proved in a similar way by replacing  $\succsim$  and  $\succsim'$  with  $\succsim''$  and  $\succsim'''$ , respectively.

(ii) Let us assume without loss of generality that  $|X| = 4$  and let  $X = \{a, b, c, d\}$  and  $\Delta_X = \{(x, x) : x \in X\}$ . Next, define  $\leq^{**} = \{(a, b), (a, c), (a, d), (b, d), (c, d)\} \cup \Delta_X$ .

It is easily checked that  $\mathcal{X}^{**} = (X, \leq^{**})$  is the Boolean lattice  $2^2$  with  $a = \top$ ,  $d = \perp$ .

Now, define the family  $\{f(x^*)\}_{x^* \in \{\perp, \top\}^N}$  as follows: for all  $x_{N \setminus \{1,2\}} \in \{\perp, \top\}^{N \setminus \{1,2\}}$

$$f(a, a, x_{N \setminus \{1,2\}}) = a, \quad f(d, d, x_{N \setminus \{1,2\}}) = d, \quad f(a, d, x_{N \setminus \{1,2\}}) = b, \quad f(d, a, x_{N \setminus \{1,2\}}) = c.$$

Then, consider the nested sequence of medians that provides the run of the median tree-automaton  $\mathcal{A}_\mu^{I, \lambda}$  as initialized with ballot profile  $x_N$  and applied to the finite  $(\Sigma^\mu, I)$ -tree  $T = T(x_N, \{f(x^*)\}_{x^* \in \{\perp, \top\}^N})$  with terminal nodes suitably labelled by projections of  $x_N$  and elements of  $\{f(x^*)\}_{x^* \in \{\perp, \top\}^N}$  as defined above (notice that  $f$  is by construction an extension of  $f'$  to  $X^N$  as mentioned above under part (i) of the present proof). A few simple if tedious calculations immediately establish that for all  $x_{N \setminus \{1,2\}} \in X^{N \setminus \{1,2\}}$ :

$$\begin{aligned} f(a, c, x_{N \setminus \{1,2\}}) &= f(b, a, x_{N \setminus \{1,2\}}) = f(b, c, x_{N \setminus \{1,2\}}) = a, \\ f(b, b, x_{N \setminus \{1,2\}}) &= f(a, b, x_{N \setminus \{1,2\}}) = f(b, d, x_{N \setminus \{1,2\}}) = b, \\ f(c, c, x_{N \setminus \{1,2\}}) &= f(c, a, x_{N \setminus \{1,2\}}) = f(d, c, x_{N \setminus \{1,2\}}) = c, \\ f(c, d, x_{N \setminus \{1,2\}}) &= f(d, c, x_{N \setminus \{1,2\}}) = f(c, b, x_{N \setminus \{1,2\}}) = d. \end{aligned}$$

By construction, and in view of Lemma 3 above,  $f$  is  $B_{\mathcal{X}^{**}}$ -monotonic. Therefore, by Lemma 1,  $f$  is also strategy-proof on  $U_{\mathcal{X}^{**}}^N$ .

Now, take

$$\begin{aligned} \succ &= \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d), (d, c)\} \cup \Delta_X, \\ \succ' &= \{(d, b), (d, c), (d, a), (b, c), (b, a), (c, a), (a, c)\} \cup \Delta_X, \end{aligned}$$

as defined in Remark 2 above.

First, observe that both  $\succ$  and  $\succ'$  are in  $U_{\mathcal{X}^{**}}^N$ , i.e. are unimodal with respect to  $\mathcal{X}^{**}$ : indeed,  $\text{top}(\succ) = a$ ,  $\text{top}(\succ') = d$  and it is immediately seen that

$$\begin{aligned} B_{\mathcal{X}}(X, \leq^{**}) &= \left\{ \begin{array}{l} (a, b, d), (a, c, d), (b, a, c), (b, d, c), (d, b, a), \\ (d, c, a), (c, a, b), (c, d, b) \end{array} \right\} \cup \\ &\cup \{(x, y, z) \in X^3 : x = y \text{ or } z = y\}. \end{aligned}$$

But then, since  $\{(b, d), (c, d), (a, b), (d, c)\} \cup \Delta_X$  is a subrelation of  $\succ$  and  $\{(b, a), (c, a), (a, c), (d, c)\} \cup \Delta_X$  is a subrelation of  $\succ'$ , it follows that  $\succ$  and  $\succ'$  are also unimodal with respect to  $\mathcal{X}^{**}$ . Now, take any preference profile  $(\succ_i)_{i \in N}$  such that  $\succ_1 = \succ'$  and  $\succ_2 = \succ$ , hence  $\text{top}(\succ_1) = d$ ,  $\text{top}(\succ_2) = a$ . Then, for any  $x_{N \setminus \{1,2\}} \in X^{N \setminus \{1,2\}}$ , both  $f(a, d, x_{N \setminus \{1,2\}}) \succ_1 f(\text{top}(\succ_1), \text{top}(\succ_2), x_{N \setminus \{1,2\}})$  and  $f(a, d, x_{N \setminus \{1,2\}}) \succ_2 f(\text{top}(\succ_1), \text{top}(\succ_2), x_{N \setminus \{1,2\}})$ : it follows that, again, coalition  $\{1, 2\}$  can manipulate the outcome at  $(\succ_i)_{i \in N}$  namely  $f$  is *not* coalitionally strategy-proof.

Again, strategy-proofness and failure of coalitional strategy-proofness of  $f$  on  $U_{\mathcal{X}^{**}}^N$  follows from the very same argument, by positing  $\succ_1 = \succ''$  and  $\succ_2 = \succ'''$ .  $\square$

*Proof of Corollary 1.* (i)  $\implies$  (ii) It follows immediately from Theorem 2 (ii) above;

(ii)  $\implies$  (i) For the case concerning  $U_Y^N$ , the statement follows from a straightforward extension and adaptation of the proof of Proposition 4 of Danilov (1994) concerning voting rules on unimodal domains of *linear orders* in *undirected bounded trees* (details available from the authors upon request), and is indeed already stated without explicit proof in Moulin (1980). As far as  $S_Y^N$  is concerned, the statement follows e.g. from Theorem 2 and Proposition 3 of Barberà, Berga and Moreno (2010).  $\square$

*Proof of Theorem 3.* Let us assume that on the contrary there exists a voting rule  $f : X^N \rightarrow X$  which is anonymous, locally JI-neutral on  $Y$ , locally sovereign on  $Y$ , and coalitionally strategy-proof on  $U_{\mathcal{X}}^N$  (on  $S_{\mathcal{X}}^N$ , respectively). By Theorem 1, it follows that there exists an order filter  $\mathcal{F}$  of  $(\mathcal{P}(N), \subseteq)$  such that

$$f(x_N) = \vee_{S \in \mathcal{F}} ((\wedge_{i \in S} x_i) \wedge y_S^*) \text{ for all } x_N \in X^N.$$

To begin with, observe that coalitional strategy-proofness and local sovereignty on  $Y$  jointly imply local idempotence on  $Y$  (indeed, suppose there exists  $u \in Y$ ,  $u \neq f(u^N)$ ; of course, by local sovereignty there exists  $x_N \in X^N$  such that  $f(x_N) = u$ . But then  $f$  is coalitionally manipulable at any preference profile  $(\succsim_i)_{i \in N} \in U_{\mathcal{X}}^N$  ( $(\succsim_i)_{i \in N} \in S_{\mathcal{X}}^N$ , respectively) such that  $\text{top}(\succsim_i) = u$  for all  $i \in N$ , a contradiction).

Next, for any  $u \in Y$  denote by  $S_u$  the set of all minimal coalitions  $T \in \mathcal{F}$  such that  $u \leq f(u^T, w^{N \setminus T})$  for all  $w^{N \setminus T} \in X^{N \setminus T}$ . By local idempotence of  $f$  on  $Y$ ,  $S_u \neq \emptyset$ . By anonymity of  $f$ ,  $|T| = |T'| = n_u$  for all  $T, T' \in S_u$ , and  $y_S^* = y_{S'}^* = y_u^*$  for any  $S, S' \in \mathcal{F}$  such that  $|S| = |S'| = s$ . Moreover, since by Theorem 1 coalitional strategy-proofness entails in particular  $B_{\mathcal{X}}$ -monotonicity, it also follows -by definition of  $B_{\mathcal{X}}$ -monotonicity- that for any  $i \in N \setminus T$

$$u = u \wedge f(u^T, w^{N \setminus T}) \leq f((u^{T \cup \{i\}}, w^{N \setminus (T \cup \{i\})}) \leq u \vee f(u^T, w^{N \setminus T})$$

whence, by repeated application of that argument

$$u \leq f(u^{T'}, w^{N \setminus T'}) \text{ for any } T' \subseteq N \text{ such that } |T'| \geq n_u.$$

Also, by local JI-neutrality on  $Y$  of  $f$ ,  $n_x = n_z = q$ .

Four cases are to be distinguished according to the sign of  $(q - n/2)$  and the parity of  $n$ .

( $\alpha$ ): Let us first suppose that  $q \leq n/2$ .

Then, in order to address the unimodal case consider the following triple of preference relations:

$$\succsim^* := [x \succ^* 0 \succ^* x \vee z \sim^* z \sim^* w \text{ for all } w \in X \setminus Y],$$

$$\succsim^{**} := [z \succ^{**} 0 \succ^{**} x \vee z \sim^{**} x \sim^{**} w \text{ for all } w \in X \setminus Y],$$

$$\succsim^{***} := [0 \succ^{***} x \sim^{***} z \sim^{***} x \vee z \sim^{***} w \text{ for all } w \in X \setminus Y].$$

Notice that by construction such preferences are unimodal with respect to  $\mathcal{X}$ , i.e.  $\{\succsim^*, \succsim^{**}, \succsim^{***}\} \subseteq U_{\mathcal{X}}^N$ .

Two subcases are distinguished according to the parity of  $n$ , namely

(i)  $n = 2k + 1$  for some positive integer  $k$ , and (ii)  $n = 2k$  for some positive integer  $k$ .

If ( $\alpha$ (i)) obtains then take preference profile

$$\succsim_{[O]} = ((\succsim_i^*)_{i \in \{1, \dots, k\}}, (\succsim_i^{**})_{i \in \{k+1, \dots, 2k\}}, \succsim_{2k+1}^{***})$$

and compute  $f(y_N) = \vee_{S \in \mathcal{F}} ((\wedge_{i \in S} y_i) \wedge y_S^*)$

where  $y_N = \text{top}(\succsim_{[O]})$  i.e.  $y_i = x$  for all  $i \in \{1, \dots, k\}$ ,  $y_i = z$  for all  $i \in \{k+1, \dots, 2n\}$ , and  $y_{2k+1} = 0$ .

By construction,  $f(y_N)$  is the l.u.b. of a nonempty family  $\mathcal{T}$  of terms belonging to some of the following jointly exhaustive, partially overlapping classes:

$$T_1 = \{\wedge_{j \in J} v_j : J \text{ is a finite set } J \text{ and there exists } j \in J \text{ such that } v_j = 0\},$$

$$T_2 = \{\wedge_{j \in J} v_j : J \text{ is a finite set } J \text{ and there exist } j, h \in J \text{ such that } v_j = x \text{ and } v_h = z\},$$

$$T_3 = \{\wedge_{j \in J} v_j : J \text{ is a finite set and there exists } J' \subseteq J \text{ such that } |J'| \geq q \text{ and } v_j = x \text{ for all } j \in J'\},$$

$$T_4 = \{\wedge_{j \in J} v_j : J \text{ is a finite set and there exists } J' \subseteq J \text{ such that } |J'| \geq q \text{ and } v_j = z \text{ for all } j \in J'\}.$$

Moreover,  $t = \wedge_{j \in J} v_j = 0$  for all  $t \in T_1 \cup T_2$  hence, by construction,  $T_3 \cap \mathcal{T} \neq \emptyset \neq T_4 \cap \mathcal{T}$ . On the other hand,  $t_3 \geq x$  and  $t_4 \geq z$  for any  $t_3 \in T_3$  and  $t_4 \in T_4$ .

It follows that  $f(y_N) \geq x \vee z$ .

If  $(\alpha(ii))$  obtains then take preference profile

$$\succ_{[E]} = ((\succ_i^*)_{i \in \{1, \dots, k\}}, (\succ_i^{**})_{i \in \{k+1, \dots, 2k\}}),$$

and compute  $f(y'_N) = \vee_{S \in \mathcal{F}} ((\wedge_{i \in S} y'_i) \wedge y_s^*)$ ,

where  $y'_N = \text{top}(\succ_{[E]})$  i.e.  $y'_i = x$  for all  $i \in \{1, \dots, k\}$ , and  $y'_i = z$  for all  $i \in \{k+1, \dots, 2n\}$ ,

Again,  $f(y'_N)$  is the l.u.b. of a nonempty family  $\mathcal{T}$  of terms belonging to some of the following jointly exhaustive, partially overlapping classes:

$$T'_1 = \{\wedge_{j \in J} v_j : J \text{ is a finite set } J \text{ and there exist } j, h \in J \text{ such that } v_j = x \text{ and } v_h = z\},$$

$$T'_2 = \{\wedge_{j \in J} v_j : J \text{ is a finite set and there exists } J' \subseteq J \text{ such that } |J'| \geq q \text{ and } v_j = x \text{ for all } j \in J'\},$$

$$T'_3 = \{\wedge_{j \in J} v_j : J \text{ is a finite set and there exists } J' \subseteq J \text{ such that } |J'| \geq q \text{ and } v_j = z \text{ for all } j \in J'\}.$$

Moreover,  $t = \wedge_{j \in J} v_j = 0$  for all  $t \in T'_1$  hence, by construction,  $T'_2 \cap \mathcal{T} \neq \emptyset \neq T'_3 \cap \mathcal{T}$ . On the other hand,  $t_2 \geq x$  and  $t_3 \geq z$  for any  $t_2 \in T'_2$  and  $t_3 \in T'_3$ .

It follows, again, that  $f(y'_N) \geq x \vee z$ .

Now, take  $u_N \in X^N$  with  $u_i = 0$  for all  $i \in N$ : by local idempotence,  $f(u_N) = 0$ .

Thus, if  $n = 2k+1$ ,  $f((u_i = 0)_{i \in N \setminus \{2k+1\}}, y_{2k+1} = 0) = f(u_N) \succ_i f(y_N)$  for all  $i \in N \setminus \{2k+1\}$ .

Similarly, if  $n = 2k$ , then  $f(u_N) \succ_i f(y_N)$  for all  $i \in N$ . Hence,  $f$  is coalitionally manipulable at unimodal preference profile  $\succ^{[O]}$  (at unimodal preference profile  $\succ^{[E]}$ , respectively), a contradiction.

The locally strictly unimodal case can be addressed precisely by the same argument, provided preference profile  $(\succ^*, \succ^{**}, \succ^{***})$  is replaced by any locally strictly unimodal preference profile  $(\succ', \succ'', \succ''')$  such that

$$\succ' := [x \succ' 0 \succ' x \vee z \succ' z \succ' w \text{ for all } w \in X \setminus Y],$$

$$\succ'' := [z \succ'' 0 \succ'' x \vee z \succ'' x \succ'' w \text{ for all } w \in X \setminus Y],$$

$$\succ''' := [0 \succ''' x \succ''' z \succ''' x \vee z \succ''' w \text{ for all } w \in X \setminus Y].$$

( $\beta$ ) Let us now assume that, on the contrary,  $q > (n/2)$ .

Then, consider the following triple of preference relations:

$$\succ^\circ := [x \succ^\circ x \vee z \succ^\circ 0 \sim^\circ z \sim^\circ w \text{ for all } w \in X \setminus Y],$$

$$\succ^{\circ\circ} := [z \succ^{\circ\circ} x \vee z \succ^{\circ\circ} 0 \sim^{\circ\circ} x \sim^{\circ\circ} w \text{ for all } w \in X \setminus Y],$$

$$\succ^{\circ\circ\circ} := [0 \succ^{\circ\circ\circ} x \sim^{\circ\circ\circ} z \sim^{\circ\circ\circ} x \vee z \sim^{\circ\circ\circ} w \text{ for all } w \in X \setminus Y].$$

Notice that by construction such preferences are unimodal with respect to  $\mathcal{X}$ , i.e.  $\{\succ^\circ, \succ^{\circ\circ}, \succ'\} \subseteq U_{\mathcal{X}}$ .

Two subcases are distinguished again according to the parity of  $n$ , namely

(i)  $n = 2k+1$  for some positive integer  $k$ , and (ii)  $n = 2k$  for some positive integer  $k$ .

If  $(\beta(i))$  obtains, then take preference profile

$$\succ_{[O]}^\circ = ((\succ_i^\circ)_{i \in \{1, \dots, k\}}, (\succ_i^{\circ\circ})_{i \in \{k+1, \dots, 2k\}}, \succ_n^{\circ\circ\circ})$$

and compute  $f(w_N) = \vee_{S \in \mathcal{F}} ((\wedge_{i \in S} w_i) \wedge y_s^*)$

where  $w_N = \text{top}(\succ_{[O]}^\circ)$  i.e.  $w_i = x$  for all  $i \in \{1, \dots, k\}$ ,  $w_i = z$  for all  $i \in \{k+1, \dots, 2k\}$ , and  $w_n = 0$ .

By construction,  $f(w_N)$  is the l.u.b. of a nonempty family  $\mathcal{T}$  of terms belonging to some of the following jointly exhaustive, partially overlapping classes:

$$T_1 = \{\wedge_{j \in J} v_j : J \text{ is a finite set } J \text{ and there exists } j \in J \text{ such that } v_j = 0\},$$

$$T_2 = \{\wedge_{j \in J} v_j : J \text{ is a finite set } J \text{ and there exist } j, h \in J \text{ such that } v_j = x \text{ and } v_h = z\},$$



$$T_3 = \left\{ \begin{array}{l} \wedge_{j \in J} v_j : J \text{ is a finite set and there exists} \\ \text{a nonempty } J' \subseteq J \text{ such that } |J'| \leq k < q \text{ and } v_j = x \text{ for all } j \in J' \end{array} \right\},$$

$$T_4 = \left\{ \begin{array}{l} \wedge_{j \in J} v_j : J \text{ is a finite set and there exists} \\ \text{a nonempty } J' \subseteq J \text{ such that } |J'| \leq k < q \text{ and } v_j = z \text{ for all } j \in J' \end{array} \right\}.$$

Notice that, again,  $t = \wedge_{j \in J} v_j = 0$  for all  $t \in T_1 \cup T_2$ . Moreover, by construction,  $t = \wedge_{j \in J} v_j < x$  for all  $t \in T_3$  and  $t = \wedge_{j \in J} v_j < z$  for all  $t \in T_4$ . Since both  $x$  and  $y$  are atoms of  $\mathcal{X}$ , it follows that  $t = \wedge_{j \in J} v_j = 0$  for all  $t \in T_3 \cup T_4$  whence  $f(w_N) = 0$ .

If  $(\beta(ii))$  obtains then take preference profile

$$\succsim_{[E]}^\circ = ((\succsim_i^\circ)_{i \in \{1, \dots, k\}}, (\succsim_i^{\circ\circ})_{i \in \{k+1, \dots, 2k-1\}}, \succsim_n^{\circ\circ\circ}),$$

and compute  $f(w'_N) = \vee_{S \in \mathcal{F}} ((\wedge_{i \in S} w'_i) \wedge y_s^*)$ ,

where  $w'_N = \text{top}(\succsim'_{[E]})$  i.e.  $w'_i = x$  for all  $i \in \{1, \dots, k\}$ ,  $w'_i = z$  for all  $i \in \{k+1, \dots, 2k-1\}$ , and  $w'_n = 0$ .

Again,  $f(w'_N)$  is the l.u.b. of a nonempty family  $\mathcal{T}$  of terms belonging to some of the following jointly exhaustive, partially overlapping classes:

$$T'_1 = \{\wedge_{j \in J} v_j : J \text{ is a finite set } J \text{ and there exists } j \in J \text{ such that } v_j = 0\},$$

$$T'_2 = \left\{ \begin{array}{l} \wedge_{j \in J} v_j : J \text{ is a finite set } J \text{ and there exist} \\ j, h \in J \text{ such that } v_j = x \text{ and } v_h = z \end{array} \right\},$$

$$T'_3 = \left\{ \begin{array}{l} \wedge_{j \in J} v_j : J \text{ is a finite set and there exists} \\ \text{a nonempty } J' \subseteq J \text{ such that } |J'| < q \text{ and } v_j = x \text{ for all } j \in J' \end{array} \right\},$$

$$T'_4 = \left\{ \begin{array}{l} \wedge_{j \in J} v_j : J \text{ is a finite set and there exists} \\ \text{a nonempty } J' \subseteq J \text{ such that } |J'| < q \text{ and } v_j = z \text{ for all } j \in J' \end{array} \right\}.$$

Notice that  $t = \wedge_{j \in J} v_j = 0$  for all  $t \in T'_1$ , and for all  $t \in T'_2$  as well since  $x \wedge z = 0$ . Moreover, since  $f(w'_N) = \vee_{S \in \mathcal{F}} ((\wedge_{i \in S} w'_i) \wedge y_s^*)$ , it also follows that  $t = \wedge_{j \in J} v_j < x$  for all  $t \in T'_3 \cap \mathcal{T}$  and  $t = \wedge_{j \in J} v_j < z$  for all  $t \in T'_4 \cap \mathcal{T}$ . On the other hand,  $t_2 \geq x$  and  $t_3 \geq z$  for any  $t_2 \in T'_2$  and  $t_3 \in T'_3$ .

It follows, again, that  $f(w'_N) = 0$ .

Now, take  $u'_N \in X^N$  with  $u'_i = x \vee z$  for all  $i \in \{1, \dots, n-1\} = N \setminus \{n\}$ , and  $u'_n = 0$ . By construction,

$$f(u'_N) = \vee_{S \in \mathcal{F}} ((\wedge_{i \in S} u'_i) \wedge y_s^*)$$

is the l.u.b. of a nonempty family  $\mathcal{T}$  of terms belonging to some of the following jointly exhaustive, partially overlapping classes:

$$T''_1 = \{\wedge_{j \in J} v_j : J \text{ is a finite set } J \text{ and there exists } j \in J \text{ such that } v_j = 0\},$$

$$T''_2 = \left\{ \begin{array}{l} \wedge_{j \in J} v_j : J \text{ is a finite set and there exists} \\ \text{a nonempty } J' \subseteq J \text{ such that } |J'| < q \text{ and } v_j = x \vee z \text{ for all } j \in J' \end{array} \right\},$$

$$T''_3 = \left\{ \begin{array}{l} \wedge_{j \in J} v_j : J \text{ is a finite set and there exists} \\ J' \subseteq J \text{ such that } |J'| \geq q \text{ and } v_j = x \vee z \text{ for all } j \in J' \end{array} \right\}.$$

Observe that  $t = \wedge_{j \in J} v_j = 0$  for all  $t \in T''_1$ . Moreover, by definition of  $f$  and  $q$ , both  $y_{s'}^* < x$  and  $y_{s'}^* < z$  for all  $s' < q$ , hence  $t = \wedge_{j \in J} v_j = 0$  for all  $t \in T''_2$  as well. Furthermore,  $T''_3 \cap \mathcal{T} \neq \emptyset$  and, by definition of  $f$  and  $q$ , it must be the case that for all  $s \geq q$ , both  $x \leq y_s^*$  and  $z \leq y_s^*$  hold. Therefore,  $x \vee z \leq y_s^*$ . It follows that  $f(u'_N) = x \vee z$ .

Thus, if  $n = 2k + 1$ ,  $f(u'_N) \succsim_i f(w_N)$  for all  $i \in N \setminus \{n\}$ .

Similarly, if  $n = 2k$ , then  $f(u'_N) \succsim_i f(w'_N)$  for all  $i \in N \setminus \{n\}$ .

Hence,  $f$  is coalitionally manipulable at unimodal preference profile  $\succsim_{[O]}^\circ \in U_{\mathcal{X}}^N$  (at unimodal preference profile  $\succsim_{[E]}^\circ \in U_{\mathcal{X}}^N$ , respectively), a contradiction again, and the proof is complete.

The locally strictly unimodal case can be addressed precisely by the same argument, provided preference profile  $(\succsim^*, \succsim^{**}, \succsim^{***})$  is replaced by any locally strictly unimodal preference profile  $(\succsim^+, \succsim^{++}, \succsim^{+++})$  such that

$$\begin{aligned}\succsim^+ &:= [x \succ^+ x \vee z \succ^+ 0 \succ^+ z \succ^+ w \text{ for all } w \in X \setminus Y], \\ \succsim^{++} &:= [z \succ^{++} x \vee z \succ^{++} 0 \succ^{++} x \succ^{++} w \text{ for all } w \in X \setminus Y], \\ \succsim^{+++} &:= [0 \succ^{+++} x \succ^{+++} z \succ^{+++} x \vee z \succ^{+++} w \text{ for all } w \in X \setminus Y].\end{aligned}\quad \square$$

## 7. APPENDIX 2: TREE AUTOMATA

Tree automata are a powerful generalization of the more widely known sequential automata (see chpt. 2 of Adámek and Trnková (1990) for a thorough treatment of tree automata in a categorical framework).

A *finitary type* is a pair  $\Sigma = (S, \alpha)$  where  $S$  is a set (whose members denote operation symbols) and  $\alpha \in \mathbb{N}^S$  is a function mapping  $S$  into natural numbers which specifies for each  $s \in S$  the corresponding (finitary) ‘arity’  $\alpha(s) \in \mathbb{N}$  of the corresponding operation.<sup>26</sup> A  $\Sigma$ -*algebra* is a pair  $A = (X, \{f_s\}_{s \in S})$  where  $X$  is a set and, for each  $s \in \Sigma$ ,  $f_s : X^{\alpha(s)} \rightarrow X$  is an  $\alpha(s)$ -ary *operation* on  $X$ . For any pair of  $\Sigma$ -algebras  $A = (X, \{f_s\}_{s \in S})$ ,  $B = (X', \{f'_s\}_{s \in S})$  a *homomorphism* of  $A$  into  $B$  is an operation-preserving function  $\varphi : X \rightarrow X'$ , namely for each  $s \in S$  and  $x_1, \dots, x_{\alpha(s)} \in X$ ,  $f'_s(\varphi(x_1), \dots, \varphi(x_{\alpha(s)})) = \varphi(f_s(x_1, \dots, x_{\alpha(s)}))$ .

A *non-initial  $\Sigma$ -tree automaton* is a quadruple  $\mathcal{A} = (Q, \{d_s\}_{s \in S}, Y, h)$  where  $Q$  is a set, the *set of states*,  $d_s : Q^{\alpha(s)} \rightarrow Q$  is  $\alpha(s)$ -ary *operation* on  $Q$  for any  $s \in S$ ,  $Y$  is a set, the *output alphabet*, and  $h : Q \rightarrow Y$  is the *output function*: thus,  $\mathcal{A}$  amounts to a  $\Sigma$ -algebra  $(Q, \{d_s\}_{s \in S})$  supplemented with an output alphabet and an output function modeling the ‘external’ effects or observable behaviour of the former.

A (initial)  $\Sigma$ -*tree automaton* is a sextuple  $\mathcal{A}^{I, \lambda} = (Q, \{d_s\}_{s \in S}, Y, h, I, \lambda)$  where

$\mathcal{A} = (Q, \{d_s\}_{s \in S}, Y, h)$  is a non-initial  $\Sigma$ -tree automaton,  $I$  is a set, the *set of variables*, and  $\lambda : I \rightarrow Q$  is the *initialization function*.

For any set  $I$  of variables, a *finite labelled  $(\Sigma, I)$ -tree* is a triple  $T = (P, \leq, p_0)$  such that: (i)  $P \subseteq \Sigma \cup I$  is the *finite set of nodes*, (ii)  $\leq$  is a *partial order on  $P$  with the tree property* namely for any  $p \in P$  the set  $p \downarrow = \{q \in P : q \leq p\}$  of  $\leq$ -predecessors of  $p$  is linearly ordered i.e. is a chain, (iii)  $p_0 \in P$  is the *root* of  $\mathcal{T}$  i.e. the minimum of  $(P, \leq)$ , (iv) for any  $p \in P$  if  $p \in I$  or  $p = s$  for some  $s \in S$  such that  $\alpha(s) = 0$  then  $p$  is  $\leq$ -maximal (or a *terminal node* of  $T$ ); (v) for any  $p \in P$  if  $p = s \in \Sigma$  then  $p$  is the *lower cover* (or immediate  $\leq$ -predecessor) of precisely  $\alpha(s)$  nodes.

Observe that any  $p \in P$  induces a finite labelled  $(\Sigma, I)$ -tree  $(p \uparrow, \leq|_{p \uparrow}, p)$ , the *sub- $(\Sigma, I)$ -tree of  $T$  with root  $p$*  (where  $p \uparrow = \{q \in P : p \leq q\}$ ). In particular, each terminal node  $p$  may be identified with a *degenerate* one-node finite labelled  $(\Sigma, I)$ -tree  $T_p = (\{p\}, =, p)$ .

The  $\mathcal{A}^{I, \lambda}$ -*initialized* version of a finite labelled  $(\Sigma, I)$ -tree  $T$ , denoted by  $T(\mathcal{A}^{I, \lambda})$ , is obtained from  $T$  by substituting state  $\lambda(p) \in Q$  for each variable  $p \in P \cap I$ .

<sup>26</sup>The *degree*  $m = m(\Sigma)$  of finitary type  $\Sigma$  is the largest ‘arity’ of an operation denoted by one of its symbols i.e.  $m(\Sigma) = \vee_{s \in S} \alpha(s)$  (if such ‘arities’ are unbounded posit  $m(\Sigma) = \omega$  where  $\omega = |\mathbb{N}|$ ). As usual,  $m$  is identified here with the set of all natural numbers smaller than  $m$ , starting with 0.

The set of all finite labelled  $(\Sigma, I)$ -trees is denoted  $\mathcal{T}_I$  and can be naturally endowed with the structure of a  $\Sigma$ -algebra by positing for any  $s \in S$ ,  $\psi_s : \mathcal{T}_I^{\alpha(s)} \rightarrow \mathcal{T}_I$  defined as follows: for each  $T_1, \dots, T_{\alpha(s)} \in \mathcal{T}_I$ ,

$\psi_s(T_1, \dots, T_{\alpha(s)})$  is the finite labelled  $(\Sigma, I)$ -tree having  $s$  as its root, immediately followed by trees  $T_1, \dots, T_{\alpha(s)}$  themselves. Observe that each  $z \in I$  can be identified with a trivial one-node tree  $t_z$  in  $\mathcal{T}_I$ .

Moreover, it can be easily checked that  $(\mathcal{T}_I, \{\psi_s\}_{s \in S})$  is in fact *the free  $\Sigma$ -algebra generated by  $I$* , namely for each  $\Sigma$ -algebra  $(Q, \{d_s\}_{s \in S})$  and for each function  $\lambda : I \rightarrow Q$  there exists a unique homomorphism  $\rho : \mathcal{T}_I \rightarrow Q$  of  $(\mathcal{T}_I, \{\psi_s\}_{s \in S})$  into  $(Q, \{d_s\}_{s \in S})$  extending  $\lambda$  to the entire set  $\mathcal{T}_I$  of finite labelled  $(\Sigma, I)$ -trees.

A  $\Sigma$ -tree automaton  $\mathcal{A}^{I, \lambda} = (Q, \{d_s\}_{s \in S}, Y, h, I, \lambda)$  acts on a finite labelled  $(\Sigma, I)$ -tree  $T$  by initializing it through  $\lambda$ , computing the value at  $T(\mathcal{A}^{I, \lambda})$  of the *run map* of  $\mathcal{A}^{I, \lambda}$  i.e. the unique homomorphism  $\rho : \mathcal{T}_I \rightarrow Q$  of  $(\mathcal{T}_I, \{\psi_s\}_{s \in S})$  into  $(Q, \{d_s\}_{s \in S})$  extending  $\lambda$ , and taking  $h(\rho(T(\mathcal{A}^{I, \lambda})))$  as the output of  $\Sigma$ -tree automaton  $\mathcal{A}^{I, \lambda}$  when applied to finite labelled  $(\Sigma, I)$ -tree  $T$ . Hence, the action of  $\mathcal{A}^{I, \lambda}$  on  $\mathcal{T}_I$  is summarized by the *behaviour map* of  $\mathcal{A}^{I, \lambda}$ , namely  $\mathcal{A}^{I, \lambda} = h \circ \rho : \mathcal{T}_I \rightarrow Y$ .

That computation, namely  $\mathcal{A}^{I, \lambda}(T)$  can also be described as a finite nested sequence  $(T^{(i)} = (P^{(i)}, \leq^{(i)}, p_0^{(i)})_{i=0, \dots, k})$  of finite labelled  $\Sigma$ -trees denoting the steps of a backward induction algorithm, namely

- (i)  $T^{(0)} = T$ ,  $T^{(1)} = T(\mathcal{A}^{I, \lambda})$ ,  $T^{(k)} = T_q = (\{q\}, =, q)$  for some  $q \in Q$ , and
- (ii) for any  $i = 1, \dots, k-1$ ,  $P^{(i+1)} \subseteq P^{(i)}$ ,  $\leq^{(i+1)} = \leq^{(i)} \cap (P^{(i+1)} \times P^{(i+1)})$ , and  $T^{(i+1)}$  is obtained from  $T^{(i)}$  by replacing a non-terminal node labelled by some operation symbol  $s \in S$  (having only terminal nodes labelled  $q_1, \dots, q_{\alpha(s)}$  as immediate  $\leq$ -successors) with a new terminal node labelled with state  $\delta_s(q_1, \dots, q_{\alpha(s)})$ .

Notice, however, that since those trees amount to sub- $(\Sigma, I)$ -trees of  $T$  and can therefore be identified with their roots, it follows that  $\mathcal{A}^{I, \lambda}(T) = h(d_{p_0}(\dots(d_s(q_1, \dots, q_{\alpha(s)}))\dots))$  where  $s = p$  is the immediate  $\leq$ -predecessor of some terminal node,  $q_1 = \lambda(z_1), \dots, q_{\alpha(s)} = \lambda(z_{\alpha(s)})$  for some  $z_1, \dots, z_{\alpha(s)} \in I$ , namely  $\mathcal{A}^{I, \lambda}(T)$  -the *behaviour* of  $\mathcal{A}^{I, \lambda}$  at  $T$ - can also be equivalently written as the *output*-value of the outcome of a nested sequence of  $\Sigma$ -operations dictated by  $T$  and applied to  $\lambda(I)$  that detail the computation steps of the *run map* of  $\mathcal{A}^{I, \lambda}$  at  $T$ .

## REFERENCES

- [1] Adámek J., V. Trnková (1990): *Automata and algebras in categories*. Kluwer, Dordrecht.
- [2] Balinski M., R. Laraki (2010): *Majority judgment. Measuring, ranking, and electing*. MIT Press, Cambridge MA.
- [3] Bandelt H.J., J. Hedlíková (1983): Median algebras, *Discrete Mathematics* 45, 1-30.
- [4] Barberà S., D. Berga, B. Moreno (2010): Individual versus group strategy-proofness: When do they coincide?, *Journal of Economic Theory* 145, 1648-1674.
- [5] Barberà S., F. Gul, E. Stacchetti (1993): Generalized median voter schemes and committees, *Journal of Economic Theory* 61, 262-289.
- [6] Barberà S., J. Massò, A. Neme (1997): Voting under constraints, *Journal of Economic Theory* 76, 298-321.
- [7] Barberà S., H. Sonnenschein, L. Zhou (1991): Voting by committees, *Econometrica* 59, 595-609.
- [8] Birkhoff G., S.A. Kiss (1947): A ternary operation in distributive lattices, *Bulletin of the American Mathematical Society* 53, 749-752.
- [9] Black D. (1948): On the rationale of group decision making, *Journal of Political Economy* 56, 23-34.

- [10] Border K.C., J.S. Jordan (1983): Straightforward elections, unanimity and phantom voters, *Review of Economic Studies* 50, 153-170.
- [11] Bordes G., G. Laffond, M. Le Breton (2012): Euclidean preferences, option sets and strategy proofness, IDEI Working Paper Series 717, Toulouse.
- [12] Bossert W., Y. Sprumont (2014): Strategy-proof preference aggregation: possibilities and characterizations, Mimeo, University of Montreal.
- [13] Bossert W., K. Suzumura (2010): *Consistency, choice, and rationality*. Harvard University Press, Cambridge MA.
- [14] Brown D.J., S.A. Ross (1991): Spanning, valuation and options, *Economic Theory* 1, 3-12.
- [15] Chatterji S., R. Sanver, A. Sen (2013): On domains that admit well-behaved strategy-proof social choice functions, *Journal of Economic Theory* 148, 1050-1073.
- [16] Chichilnisky G., G. Heal (1997): The geometry of implementation: a necessary and sufficient condition for straightforward games, *Social Choice and Welfare* 14, 259-294.
- [17] Ching S. (1997): Strategy-proofness and “median voters”, *International Journal of Game Theory* 26, 473-490.
- [18] Danilov V.I. (1994): The structure of non-manipulable social choice rules on a tree, *Mathematical Social Sciences* 27, 123-131.
- [19] Danilov V.I., A.I. Sotskov (2002): *Social choice mechanisms*. Springer, Berlin.
- [20] Demange G. (1982): Single-peaked orders on a tree, *Mathematical Social Sciences* 3, 389-396.
- [21] Glivenko V. (1936): Géométrie des systèmes de choses normées, *American Journal of Mathematics* 58, 799-828.
- [22] Grandmont J.-M. (1978): Intermediate preferences and the majority rule, *Econometrica* 46, 317-330.
- [23] Isbell J.R. (1980): Median algebra, *Transactions of the American Mathematical Society* 260, 319-362.
- [24] Le Breton M., V. Zaporozhets (2009): On the equivalence of coalitional and individual strategy-proofness properties, *Social Choice and Welfare* 33, 287-309.
- [25] Monjardet B. (1990): Arrowian characterizations of latticial federation consensus functions, *Mathematical Social Sciences* 20, 51-71.
- [26] Moulin H. (1980): On strategy-proofness and single peakedness, *Public Choice* 35, 437-455.
- [27] Nehring K., C. Puppe (2007 (a)): The structure of strategy-proof social choice - Part I: General characterization and possibility results on median spaces, *Journal of Economic Theory* 135, 269-305.
- [28] Nehring K., C. Puppe (2007 (b)): Efficient and strategy-proof voting rules: a characterization, *Games and Economic Behavior*, 59, 132-153.
- [29] Peremans W., H. Peters, H. van der Stel, T. Storcken (1997): Strategy-proofness on Euclidean spaces, *Social Choice and Welfare* 14, 379-401.
- [30] Peters H., H. van der Stel, T. Storcken (1992): Pareto optimality, anonymity, and strategy-proofness in location problems, *International Journal of Game Theory* 21, 221-235.
- [31] Schummer J., R.V. Vohra (2002): Strategy-proof location on a network, *Journal of Economic Theory* 104, 405-428.
- [32] Sholander M. (1952): Trees, lattices, order, and betweenness, *Proceedings of the American Mathematical Society* 3, 369-381.
- [33] Sholander M. (1954(a)): Medians and betweenness, *Proceedings of the American Mathematical Society* 5, 801-807.
- [34] Sholander M. (1954(b)): Medians, lattices, and trees, *Proceedings of the American Mathematical Society* 5, 808-812.
- [35] Vannucci S. (2012): Unimodality and equivalence of simple and coalitional strategy-proofness in convex idempotent interval spaces. Quad. DEPS 668, Università di Siena.